

M.A. Evgrafov

Asymptotic Estimates and Entire Functions

Asymptotic Estimates and Entire Functions

RUSSIAN TRACTS ON ADVANCED MATHEMATICS AND PHYSICS

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Asymptotic Estimates and Entire Functions

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Translator's Preface

Asymptotic estimates are of great importance in mathematical analysis and its applications. Few books have appeared in English dealing with this subject, and the present volume is a valuable addition to the literature. It is not a compendium of results. Rather, it attempts to explain certain general methods by working out a number of examples.

The first chapter introduces the principal methods to be used, and the second chapter discusses the theory of entire functions of finite order. The third chapter contains the deepest results of the book: the application of the methods of chapter one to the functions of chapter two. Here the reader will find much material that has not previously appeared in book form.

This English edition is an unaltered translation of the Russian edition.

Allen Shields

October, 1961

Introduction

It would be difficult to exaggerate the importance of asymptotic estimates in all domains of analysis. Asymptotic formulae come to the aid of the investigator whenever his calculations become too complicated or even impossible. The special value of asymptotic formulae lies in the fact that they become the more accurate, the more complicated is the direct calculation of the object of these formulae. This thought was beautifully expressed by Laplace, speaking of his asymptotic formula for integrals of the form $\int_b^a \varphi(x)[f(x)]^n dx$:

"... the greater is the need for this method, the more accurate it becomes..." In any investigation whose purpose is to disclose the law of behaviour of some object near a parameter value where this object becomes excessively complicated, we always begin by attempting to replace our complicated object by a simpler one with the same basic properties, that is, in essence, by finding an asymptotic formula. It can be said without exaggeration that the success of an investigation always depends on finding a good asymptotic formula, that is, one which preserves the basic properties being studied but which has been simplified by eliminating second order properties. Because of the very wide domain of application of asymptotic formulae, it is natural to find a very great diversity in the methods of obtaining them. Most asymptotic estimates are obtained by the most primitive methods, using both essential and inessential properties of the problem. This is why there are many works on asymptotic properties of special objects, such

as, for example, asymptotic properties of the solutions of differential equations, asymptotic properties of eigenvalues, etc., but almost no works on general methods of obtaining asymptotic estimates. But such methods exist, and they can be systematized and generalized. Of course, not every problem that can be solved by special methods can also be solved by general methods, but most of them can, and often the solution given by the general methods is simpler and more complete.

The theory of entire functions is very closely related to asymptotic estimates. One reason for this is that all (or nearly all) problems in the theory of entire functions consist in establishing connections between the behaviour of the function in a neighborhood of infinity and its other properties. Since here one is interested in the law of behaviour for large values of the argument, it is clear that the role of asymptotic estimates will be very great, if not in the presentation of the theory, then at least in the success of the investigations. Another reason is that in many questions of analysis the objects of the asymptotic estimates are entire functions, or functions that are easily expressed in terms of entire functions. This includes, for example, all special functions. Therefore, having in mind possible applications, the theory of entire functions must be developed in such a way that its results can be used to obtain asymptotic estimates when applied to concrete entire functions.

In this book we attempt to present the basic material of the theory of entire functions in the closest possible connection with asymptotic estimates. The purpose of this is to obtain material for examples, and at the same time to enable the reader in the shortest possible time to master the theory of entire functions sufficiently to enable him to apply it to problems of interest to himself.

Contents

Translator's Preface.....	v
Introduction.....	vii

Chapter I

Methods of obtaining asymptotic estimates

§1. The simplest examples of asymptotic estimates and the concept of an asymptotic series.....	1
§2. The Euler-Maclaurin summation formula.....	9
§3. The Laplace method for asymptotic estimation of integrals.....	17
§4. The method of generating functions.....	28
§5. The saddlepoint method.....	42

Chapter II

Basic material from the theory of entire functions

§1. The concept of the order of growth.....	51
§2. The connection between the rate of growth of an entire function and the rate of decrease of its coefficients.....	57
§3. The connection between the rate of growth of an entire function and the number of zeros.....	69
§4. Entire functions with regularly distributed zeros....	82
§5. The Phragmen-Lindelof principle and the growth of an entire function in different directions.....	95
§6. The connection between the indicator function and the distribution of zeros.....	105
§7. The Borel transform.....	119

*Chapter III***Special cases of asymptotic estimates**

§1. Estimates of entire functions of the form

$$F(z) = \int_0^{\infty} \mu(t) e^{tz} dt, \quad F(z) = \int_{-\infty}^{\infty} \mu(t) e^{tz} dt \quad \dots \quad 127$$

§2. The Poisson summation formula and estimates of

$$\text{functions of the form } F(z) = \sum_{n=0}^{\infty} \mu(n) z^n \dots\dots\dots 137$$

§3. Functions of the form

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\mu(n)}\right), \quad F(z) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{z + \mu(n)} \dots\dots 151$$

§4. Asymptotic estimates of the zeros of entire functions. 168

Bibliography 179

Index 181

CHAPTER I

Methods of Obtaining Asymptotic Estimates

1. The simplest examples of asymptotic estimates and the concept of an asymptotic series

We shall first consider some examples in which asymptotic estimates are obtained by an elementary method, one of those that were called "primitive" in the introduction. In this method we first find the form of the estimate from general considerations, and then use the method of undetermined coefficients.

1. Consider the problem of finding asymptotic estimates for the zeros, λ_n , of the function $\sin x + (1/x)$, close to πn . A first approximation is $\lambda_n \sim \pi n$. Putting $\lambda_n = \pi n + \epsilon_n$, we obtain:

$$(-1)^n \sin \epsilon_n + \frac{1}{\pi n + \epsilon_n} = 0,$$

from which we find:

$$\epsilon_n \sim \frac{(-1)^{n-1}}{\pi n}.$$

This gives us a second approximation for λ_n :

$$\lambda_n = \pi n + \frac{(-1)^{n-1}}{\pi n} + o\left(\frac{1}{n}\right).$$

Putting $\lambda_n = \pi n + [(-1)^{n-1}/\pi n] + \epsilon_n^{(1)}$, we obtain:

$$(-1)^n \sin \left[\frac{(-1)^{n-1}}{\pi n} + \epsilon_n^{(1)} \right] = - \frac{1}{\pi n + \frac{(-1)^n}{\pi n} + \epsilon_n^{(1)}}. \quad (1)$$

2 ASYMPTOTIC ESTIMATES AND ENTIRE FUNCTIONS

The right side of equation (1) can be written in the form

$$-\frac{1}{\pi n} + (-1)^n \frac{1}{\pi^3 n^3} + o\left(\frac{1}{n^3}\right), \quad (2)$$

and the left side in the form

$$\begin{aligned} (-1)^n e_n^{(1)} - \frac{1}{\pi n} - \frac{(-1)^n}{3!} \left[e_n^{(1)} + \frac{(-1)^{n-1}}{\pi n} \right]^3 + o\left(\frac{1}{n^3}\right) = \\ = (-1)^n e_n^{(1)} - \frac{1}{\pi n} + \frac{1}{6\pi^3 n^3} + o\left(\frac{1}{n^3}\right). \end{aligned} \quad (3)$$

Comparing (2) and (3) we obtain for $e_n^{(1)}$ the expression

$$e_n^{(1)} = \frac{1}{\pi^3 n^3} \left[1 - \frac{(-1)^n}{6} \right].$$

This gives the following approximation for λ_n :

$$\lambda_n = \pi n + \frac{(-1)^{n-1}}{\pi n} + \frac{1}{\pi^3 n^3} \left[1 - \frac{(-1)^n}{6} \right] + o\left(\frac{1}{n^3}\right).$$

Clearly this process can be continued indefinitely.

2. As a second example we consider the estimation for large x of the function $\mu(x)$, inverse to $x \ln x$. We have:

$$\mu(x) \ln \mu(x) = x$$

and further

$$\mu(x) = \frac{x}{\ln \mu(x)} = \frac{x}{\ln x + \ln \frac{\mu(x)}{x}}. \quad (4)$$

But from this it is clear that for sufficiently large x

$$\frac{x}{\ln x} < \mu(x) < x,$$

that is

$$\mu(x) = \frac{x}{\ln x} + O\left(\frac{x \ln \ln x}{\ln^2 x}\right). \quad (5)$$

Substituting this first approximation that has been found into formula (4), we find the following approximation:

$$\mu(x) = \frac{x}{\ln x - \ln \ln x} + O\left(\frac{x \ln \ln x}{\ln^3 x}\right). \quad (6)$$

Just as in the previous case this process can be continued indefinitely.

3. In the following example we consider an entirely different problem, to which the same method can be applied.

Consider the behavior for $x \rightarrow \infty$ of the solution of the equation

$$y'' + \left(1 + \frac{a}{x^2}\right)y = 0. \quad (7)$$

Note first that any solution of this equation is bounded as $x \rightarrow \infty$. Indeed, let $y_0(x)$ be a solution of equation (7). Then $y_0(x)$ will be a solution of the equation

$$y'' + y = -\frac{ay_0(x)}{x^2} \quad (8)$$

with the corresponding initial conditions and, consequently, can be written in the form

$$y_0(x) = A \cos(x - \alpha) - a \int_{x_0}^x \sin(x - t) \frac{y_0(t)}{t^2} dt. \quad (9)$$

If we let

$$M(x, x_0) = \max_{x_0 \leq t \leq x} |y_0(t)|,$$

then from (9) we obtain for $x_0 > a$

$$M(x, x_0) \leq |A| + |a| M(x, x_0) \left(\frac{1}{x_0} - \frac{1}{x} \right),$$

which gives:

$$M(x, x_0) \leq \frac{|A|}{1 + \frac{|a|}{x} - \frac{|a|}{x_0}}.$$

This proves that $y_0(x)$ remains bounded as $x \rightarrow \infty$.

Now put $x_0 = \infty$ in formula (9). This gives for $y(x)$ the representation

$$y(x) = A_0 \cos(x - \alpha_0) + a \int_x^\infty \sin(x - t) \frac{y(t)}{t^2} dt. \quad (10)$$

From formula (10) we obtain at once the first approximation to $y(x)$:

$$y(x) = A_0 \cos(x - \alpha_0) + o(1).$$

Putting $y(x) = A_0 \cos(x - \alpha_0) + \epsilon_1(x)$ and substituting this expression into formula (10) we find:

$$\begin{aligned} \epsilon_1(x) &= a A_0 \int_x^\infty \sin(x - t) \cos(t - \alpha_0) \frac{dt}{t^2} + o\left(\frac{1}{x}\right) = \\ &= \frac{a A_0}{2} \int_x^\infty \{\sin(x - 2t + \alpha_0) + \sin(x - \alpha_0)\} \frac{dt}{t^2} + o\left(\frac{1}{x}\right) = \\ &= \frac{a A_0}{2x} \sin(x - \alpha_0) + o\left(\frac{1}{x}\right). \end{aligned}$$

This means that the second approximation to $y(x)$ is given by:

$$y(x) = A_0 \cos(x - \alpha_0) + \frac{a A_0}{2x} \sin(x - \alpha_0) + o\left(\frac{1}{x}\right).$$

Putting

$$y(x) = A_0 \cos(x - \alpha_0) + \frac{a A_0}{2x} \sin(x - \alpha_0) + \epsilon_2(x)$$

and substituting this expression into formula (10), we find:

$$\begin{aligned} \epsilon_2(x) &= \frac{aA_0}{2} \int_x^\infty \sin(x-2t+\alpha_0) \frac{dt}{t^2} + \\ &\quad + \frac{a^2A_0}{2} \int_x^\infty \sin(x-t) \sin(t-\alpha_0) \frac{dt}{t^3} + \\ &+ o\left(\frac{1}{x^2}\right) = \frac{aA_0}{4x^2} \cos(x-\alpha_0) + \frac{a^2A_0}{8x^2} \cos(x-\alpha_0) + \\ &\quad + o\left(\frac{1}{x^2}\right) = \frac{(2a+a^2)A_0}{8x^2} \cos(x-\alpha_0) + o\left(\frac{1}{x^2}\right), \end{aligned}$$

which gives us the following approximation for $y(x)$:

$$\begin{aligned} y(x) &= A_0 \cos(x-\alpha_0) + \frac{aA_0}{2x} \sin(x-\alpha_0) + \\ &\quad + \frac{(2a+a^2)A_0}{8x^2} \cos(x-\alpha_0) + o\left(\frac{1}{x^2}\right). \end{aligned}$$

Just as in the two preceding cases this process can be continued indefinitely.

4. For our last example we consider the problem of finding asymptotic estimates for the sums

$$S_n = f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f(1),$$

where the function $f(x)$ together with its first k derivatives are continuous on the interval $[0,1]$.

We write S_n as a Stieltjes integral

$$S_n = \int_0^1 f(x) d[nx] = n \int_0^1 f(x) dx - \int_0^1 f(x) d\psi_1(nx),$$

where $\Psi_1(x) = x - [x] - 1/2$, and $[x]$ is the largest integer

less than or equal to x . Integrating by parts we have:

$$S_n = n \int_0^1 f(x) dx + \frac{1}{2} \{f(1) - f(0)\} + \int_0^1 f'(x) \psi_1(nx) dx.$$

Since $\int_0^1 \Psi_1(x) dx = 0$, we have for $n \rightarrow \infty$

$$\int_0^1 f'(x) \psi_1(nx) dx = o(1),$$

which gives us the first approximation to S_n :

$$S_n = n \int_0^1 f(x) dx + \frac{1}{2} \{f(1) - f(0)\} + o(1).$$

Define $\Psi_2(x)$ by the conditions

$$\psi_2'(x) = -\psi_1(x), \quad \int_0^1 \psi_2(x) dx = 0.$$

Then

$$S_n = n \int_0^1 f(x) dx + \frac{1}{2} \{f(1) - f(0)\} + \\ + \frac{\psi_2(0)}{n} \{f'(1) - f'(0)\} + \frac{1}{n} \int_0^1 f''(x) \psi_2(nx) dx.$$

In view of the condition $\int_0^1 \Psi_2(x) dx = 0$ this gives us the following approximation for S_n :

$$S_n = n \int_0^1 f(x) dx + \frac{1}{2} \{f(1) - f(0)\} + \\ + \frac{\psi_2(0)}{n} \{f'(1) - f'(0)\} + o\left(\frac{1}{n}\right).$$

Defining the function $\Psi_m(x)$ by the conditions

$$\psi'_m(x) = -\psi_{m-1}(x), \quad \int_0^1 \psi_m(x) dx = 0,$$

we are led to the formula

$$S_n = n \int_0^1 f(x) dx + \sum_{m=1}^k \frac{\psi_m(0)}{n^{m-1}} \{f^{(m-1)}(1) - f^{(m-1)}(0)\} + \\ + \frac{1}{n^{k-1}} \int_0^1 f^{(k)}(x) \psi_k(x) dx. \quad (11)$$

Formula (11) is closely related to the well known Euler-Maclaurin summation formula (to which Section 2 is devoted):

$$f(1) + f(2) + \dots + f(n) = \int_0^n f(x) dx + \\ C + \frac{1}{2} f(n) + \sum_{m=1}^k B_m f^{(m)}(n) + o(f^{(k)}(n)), \quad (12)$$

which is valid provided that $f^{(k+1)}(n) = o(f^{(k)}(n))$, $n \rightarrow \infty$.

These examples that we have given are entirely characteristic for most problems involving asymptotic estimates, and on the basis of them we can formulate certain conclusions.

In each of the cases that we met the process of improving the estimates could be continued indefinitely. Each successive improvement gave us one more term (less significant than the preceding ones). However we did not raise the question of the convergence of the series that were obtained in this manner. In addition, if we

were to investigate this question we would see that in most cases these series diverge. Nevertheless, the estimates that we have obtained are valuable, since they represent as clearly as possible the behaviour of this or that very complicated function near some limiting value of the parameter. Thus the value of the asymptotic estimates does not lie in the convergence of the series, but instead in the simplicity of the terms of the series and in the ability of the first terms of the series to give an estimate of the function that becomes more and more accurate, as the parameter value approaches more and more closely to its limiting value.

In the cases that we considered we expressed the asymptotic behaviour of the roots of the equation $\sin \lambda_n = -1/\lambda_n$ in terms of powers of n , of the solution of the equation $y'' + (1+a/x^2)y = 0$ in terms of trigonometric functions and powers of x , and so forth.

Series, whose partial sums give asymptotic estimates, are usually called *asymptotic series*. More precisely this is formulated as follows:

Let x be a parameter, which we shall assume to be tending to infinity, and let $q_n(x)$ be a sequence of functions having the properties:

$$q_n(x) > 0, \quad \lim_{n \rightarrow \infty} \frac{q_{n+1}(x)}{q_n(x)} = 0.$$

Suppose, further, that $F(x)$ is defined for $0 \leq x < \infty$. The series (not necessarily convergent!)

$$\sum_{k=0}^{\infty} a_k \mu_k(x)$$

will be called an *asymptotic series for $F(x)$* , if

$$1. \quad \overline{\lim}_{x \rightarrow \infty} \frac{|\mu_n(x)|}{q_n(x)} = c, \quad 0 < c < \infty;$$

$$2. \quad F(x) - \sum_{k=0}^n a_k \mu_k(x) = o(q_n(x)),$$

that is, if the difference between $F(x)$ and each finite sum grows more slowly than the smallest term in that sum. We shall indicate this by writing

$$F(x) \sim \sum_{k=0}^{\infty} a_k \mu_k(x).$$

In example 1 we had $\mu_k(x) = q_k(x) = [x]^{-2k-1}$, where $[x]$ is the largest integer less than or equal to x ; in example 3 we had

$$q_n(x) = x^{-n}, \quad \mu_n(x) = \frac{\cos\left(x - \alpha + \frac{n\pi}{2}\right)}{x^n},$$

in the Euler-Maclaurin summation formula we had $\mu_n(x) = q_n(x) = f^{(n)}([x])$.

2. The Euler-Maclaurin summation formula

The Euler-Maclaurin series, obtained from formula (12) of Section 1, is one of the best-known asymptotic series and is widely used to obtain asymptotic estimates.

We could prove formula (12) Section 1 by adding a few words to what was said in example 4 Section 1, however we shall follow a different path. One of the reasons for this is that the method used in example 4 Section 1 does not yield an expression for the constant C of formula (12). More precisely, the expression for C is in the form of a series that will be divergent in most cases.

The method that we shall use now will only establish the Euler-Maclaurin formula for analytic functions satis-

fying certain conditions. However, this restriction of its domain of applicability is fully compensated for by other advantages.¹

Let $f(z)$ be an analytic function, regular in the half-plane $\operatorname{Re} z \geq 0$ and let $|f(x+iy)| = O(e^{-(x-1/2)y})$ for $x \geq 0$.

Consider the contour integral

$$I_n = \frac{1}{2i} \int_{C_{n,\theta}} f(z) \operatorname{ctg} \pi z \, dz,$$

where $C_{n,\theta}$ is the contour formed by the rectangle $(\Theta + iR, \Theta - iR, n - iR, n + iR)$, $0 < \Theta < 1$, and the semicircle of radius ρ and center at the point $z = n$, lying outside the rectangle. By the residue theorem

$$I_n = f(1) + f(2) + \dots + f(n).$$

Now let us attempt to calculate I_n by a different method. First of all we wish to pass to the limit as $\rho \rightarrow 0$ and $R \rightarrow \infty$. For this we note that $\operatorname{ctg}(x+iy) \rightarrow -i \operatorname{sign} y$ for $|y| \rightarrow \infty$. Therefore let us add and subtract the integrals

$$\frac{1}{2i} \int_{C_{n,\theta}^-} if(z) \, dz, \quad \frac{1}{2i} \int_{C_{n,\theta}^+} if(z) \, dz,$$

where $C_{n,\theta}^-$ is the polygonal path with vertices Θ , $\Theta - iR$, $n - iR$, $n - i\rho$, and $C_{n,\theta}^+$ is the polygonal path with vertices $n + i\rho$, $n + iR$, $\Theta + iR$, Θ . Then

¹A detailed analysis of the Euler-Maclaurin formula and its proof can be found in the book of G. H. Hardy *Divergent Series*, Oxford, 1949.

$$I_n = \frac{1}{2l} \int_{C_{n,0}^+} f(z) (\operatorname{ctg} \pi z + i) dz + \frac{1}{2l} \int_{C_{n,0}^-} f(z) (\operatorname{ctg} \pi z - i) dz + \\ + \frac{1}{2l} \int_{\Gamma_{\rho,n}} f(z) \operatorname{ctg} \pi z dz - \frac{1}{2} \int_{C_{n,0}^+} f(z) dz + \frac{1}{2} \int_{C_{n,0}^-} f(z) dz.$$

Here $\Gamma_{\rho,n}$ is the semicircle of radius ρ and center $z=n$. It is not difficult to see that, as $\rho \rightarrow 0$

$$\frac{1}{2l} \int_{\Gamma_{\rho,n}} f(z) \operatorname{ctg} \pi z dz \rightarrow \frac{1}{2} f(n).$$

In addition, by Cauchy's theorem on the independence of the integral of an analytic function of the path of integration

$$\int_{C_{n,0}^+} f(z) dz = - \int_0^{n+i\rho} f(z) dz, \quad \int_{C_{n,0}^-} f(z) dz = \int_0^{n-i\rho} f(z) dz,$$

and therefore as $\rho \rightarrow 0$

$$\frac{1}{2} \int_{C_{n,0}^-} f(z) dz - \frac{1}{2} \int_{C_{n,0}^+} f(z) dz \rightarrow \int_0^n f(x) dx.$$

Further, $f(x \pm iR) = O(e^{(2\pi - \epsilon)R})$ as $R \rightarrow \infty$, and

$$\operatorname{ctg} \pi (x \pm iR) \pm i = \mp \frac{2le^{\pm 2\pi i x}}{1 - e^{-2\pi R} e^{\pm 2\pi i x}} e^{-2\pi R} = O(e^{-2\pi R}),$$

therefore for $R \rightarrow \infty$ the integrals along the parts of the contour $C_{n,0}$ parallel to the real axis tend to zero. Passing to the limit as $\rho \rightarrow 0$, $R \rightarrow \infty$, we obtain:

$$I_n = \int_0^n f(x) dx + \frac{1}{2} f(n) + \int_0^{\theta+i\infty} f(z) \frac{dz}{e^{-2\pi iz} - 1} + \\ + \int_0^{\theta-i\infty} \frac{f(z) dz}{e^{2\pi iz} - 1} + \int_0^\infty \frac{f(n+ly) - f(n-ly)}{l} \frac{dy}{e^{2\pi y} - 1}. \quad (1)$$

Formula (1) already contains the Euler-Maclaurin formula. Indeed, if we write using Taylor's formula

$$f(n+ly) = f(n) + \frac{ly}{1!} f'(n) + \frac{(ly)^2}{2!} f''(n) + \dots,$$

$$f(n-ly) = f(n) - \frac{ly}{1!} f'(n) + \frac{(ly)^2}{2!} f''(n) - \dots,$$

and subtract we obtain:

$$\frac{f(n+ly) - f(n-ly)}{l} = 2 \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} f^{(2k+1)}(n),$$

from which by formal integration we obtain:

$$I_n \sim \int_0^n f(x) dx + C(\theta) + \frac{1}{2} f(n) + \sum_{k=1}^{\infty} B_k f^{(k)}(n), \quad (2)$$

where

$$C(\theta) = \int_0^{\theta+i\infty} \frac{f(z) dz}{e^{-2\pi iz} - 1} + \int_0^{\theta-i\infty} \frac{f(z) dz}{e^{2\pi iz} - 1}, \quad (3)$$

$$B_{2m} = 0, \quad B_{2m+1} = \frac{2(-1)^n}{(2m+1)!} \int_0^\infty \frac{y^{2m+1} dy}{e^{2\pi y} - 1}. \quad (4)$$

To complete the proof of formula (2) we must show that the series (2) really is an asymptotic series, that is that

$$I_n - \int_0^n f(x) dx - C(\theta) - \frac{1}{2} f(n) - \sum_{k=1}^m B_{2k+1} f^{(2k+1)}(n) = \\ = o(f^{(2m+1)}(n)).$$

For this it is necessary to make some additional assumptions about $f(z)$. For example, if we assume that

$$\lim_{n \rightarrow \infty} \frac{f^{(k+1)}(n)}{f^{(k)}(n)} = 0,$$

$$|f^{(k)}(n + iy)| \leq M_k |f^{(k)}(n)| e^{(2\pi - \epsilon)|y|},$$

then we obtain:

$$\begin{aligned} I_n - \int_0^n f(x) dx - C(0) - \frac{1}{2} f(n) - \sum_{k=1}^{m-1} B_{2k+1} f^{(2k+1)}(n) = \\ = \frac{(-1)^m}{(2m+1)!} \int_0^\infty \frac{f^{(2m+1)}(n + i\theta_1 y)}{e^{2\pi y} - 1} y^{2m+1} dy = \\ = \frac{f^{(2m+1)}(n)}{(2m+1)!} \int_0^\infty O(e^{-\epsilon y}) y^{2m+1} dy = O(f^{(2m+1)}(n)). \end{aligned}$$

This proves that under the assumptions we have made the Euler-Maclaurin formula really is an asymptotic series. If we had used the method of example 4 Section 1, then we could have proven the formula without the second assumption, which is not very essential in any case.

Let us note in passing that formula (4) is not the only formula, nor is it the simplest formula, for the coefficients B_k (which are called Bernoulli numbers). Let us derive another (more convenient) formula.

For this purpose in the formal equation

$$\begin{aligned} I_{n+1} - I_n = f(n+1) \sim \int_n^{n+1} f(x) dx + \\ + \frac{1}{2} [f(n+1) - f(n)] + \sum_{k=1}^{\infty} B_k [f^{(k)}(n+1) - f^{(k)}(n)] \end{aligned}$$

we put $f(z) = e^{tz}$. Then we obtain:

$$e^{nt+t} = \frac{e^{nt}(e^t - 1)}{t} + \frac{1}{2} e^{nt}(e^t - 1) + \sum_{k=1}^{\infty} B_k t^k e^{nt}(e^t - 1).$$

From this we have:

$$\sum_{k=1}^{\infty} B_k t^k = \frac{1}{2} \frac{e^t + 1}{e^t - 1} - \frac{1}{t}. \quad (5)$$

This formula is the simplest way of determining the numbers B^k .²

A more complicated question in using the Euler-Maclaurin formula is the determination of the constant $C(\Theta)$. Often this task can be simplified by a suitable choice of Θ . Usually the most convenient value is $\Theta = 0$. For $\Theta = 0$ formula (3) becomes

$$C(0) = -\frac{1}{2} f(0) - \int_0^{\infty} \frac{f(ty) - f(-ty)}{t} \frac{dy}{e^{2\pi y} - 1}, \quad (6)$$

since

$$\begin{aligned} & \int_0^{\Theta+i\infty} \frac{f(z) dz}{e^{-2\pi iz} - 1} + \int_0^{\Theta-i\infty} \frac{f(z) dz}{e^{2\pi iz} - 1} = \\ &= \int_0^{\Theta+i\infty} \frac{f(z) - f(0)}{e^{-2\pi iz} - 1} dz + \int_0^{\Theta-i\infty} \frac{f(z) - f(0)}{e^{2\pi iz} - 1} dz + \\ & \quad + \frac{1}{2\pi i} \ln \frac{1 - e^{-2\pi i\Theta}}{1 - e^{2\pi i\Theta}} f(0), \end{aligned}$$

²It should be noted that nowadays it is customary to refer to certain other numbers as the Bernoulli numbers. These numbers are determined by the expansion.

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

which gives formula (6) as $\Theta \rightarrow 0$.

The integrals that occur in the expression for $C(\Theta)$ are usually closely connected with the function $\zeta(s) = \sum n^{-s}$ and can often be expressed in terms of it.

Let us consider two of the simplest examples.

1. $f(n) = n^a$, $a > 0$. By formula (2) we can write:

$$\sum_{k=1}^n k^a \sim \frac{n^{a+1}}{a+1} - \frac{\theta^{a+1}}{a+1} + C_a(\theta) + \sum_{k=1}^{\infty} B_k \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{n^{k-\alpha}}.$$

To determine $C_a(\Theta)$ we have formula (3):

$$C_a(\theta) = \int_{\theta}^{\theta+i\infty} \frac{z^a dz}{e^{-2\pi iz} - 1} + \int_{\theta}^{\theta-i\infty} \frac{z^a dz}{e^{2\pi iz} - 1}.$$

For $\Theta = 0$ we obtain:

$$\begin{aligned} C_a(0) &= -\frac{1}{i} \left(e^{\frac{\pi i a}{2}} - e^{-\frac{\pi i a}{2}} \right) \int_0^{\infty} \frac{y^a dy}{e^{2\pi y} - 1} = \\ &= -2 \sin \frac{\pi a}{2} \int_0^{\infty} \sum_{m=1}^{\infty} e^{-2\pi m y} y^a dy = \\ &= -2 \sin \frac{\pi a}{2} \sum_{m=1}^{\infty} (2\pi m)^{-a-1} \int_0^{\infty} e^{-u} u^a du = \\ &= -2\Gamma(\alpha+1)\zeta(\alpha+1)(2\pi)^{-\alpha-1} \sin \frac{\pi \alpha}{2}. \end{aligned}$$

If we use the functional equation for $\zeta(s)$ ³

$$\zeta(s) = 2\Gamma(1-s)\zeta(1-s)(2\pi)^{s-1} \sin \frac{\pi s}{2}, \quad (7)$$

³See, for example, E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford, 1951.

then this formula can be given a simpler form

$$C_a(0) = \zeta(-a).$$

When $C_a(0)$ is known, $C_a(\theta)$ can be found, since $C_a(\theta) - (\theta^{a+1})/(\alpha+1)$ does not depend on θ . That is,

$$C_a(\theta) = \zeta(-a) + \frac{\theta^{a+1}}{a+1}. \quad (8)$$

2. $f(n) = \ln n$. By formula (2) we can write:

$$\sum_{k=1}^n \ln k \sim \int_1^n \ln x \, dx + \frac{1}{2} \ln n + C(\theta) + \sum_{k=0}^{\infty} (-1)^k B_{k+1} \frac{k!}{n^{k+1}}.$$

We determine $C(\theta)$ from formula (3):

$$C(\theta) = \int_0^{1+i\infty} \frac{\ln z \, dz}{e^{-2\pi i s} - 1} + \int_0^{\theta-i\infty} \frac{\ln z \, dz}{e^{2\pi i s} - 1}.$$

Note that $C(\theta) = (d/d\alpha)C_\alpha(\theta)|_{\alpha=0}$, where $C_\alpha(\theta)$ is the constant of example 1. From this we obtain immediately:

$$C(\theta) = -\zeta'(0) + \theta \ln \theta - \theta.$$

In particular, $C(0) = -\zeta'(0)$. The value $\zeta'(0) = -\frac{1}{2} \ln 2\pi$ can be found from formula (7) with no difficulty.

Thus we finally obtain:

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \frac{1}{2} \ln 2\pi + \sum_{k=0}^{\infty} \frac{(-1)^k B_{k+1} k!}{n^{k+1}}. \quad (9)$$

This series can easily be seen to diverge, since it follows from (5) that

$$\lim_{k \rightarrow \infty} \sqrt[k]{|B_k|} = \frac{1}{2\pi}.$$

3. The Laplace method for asymptotic estimation of integrals

A still more valuable method of obtaining asymptotic estimates is the so-called *Laplace method*. This method has many more applications than does the Euler-Maclaurin formula.

The Laplace method is applied to integrals of the form

$$\int_a^b F(x, t) dt,$$

in which the function $F(x, t)$ has just one maximum inside or at an endpoint of the interval (a, b) , while this maximum becomes more pronounced as the parameter x becomes larger. Thus, for example, one of the most characteristic cases is when $F(x, t) = \varphi(t)e^{xh(t)}$, where $h(t)$ has exactly one maximum on (a, b) . The Laplace method is one of the most important components of the saddlepoint method, which is certainly one of the most powerful methods for obtaining asymptotic estimates. We shall try to work out as carefully as possible all cases where the Laplace method is applied, so that when we discuss the saddlepoint method we can concentrate our attention on those features that are new in comparison with the Laplace method.

We begin with the simplest theorems.

THEOREM 1. *Let*

$$F(x) = \int_0^a \varphi(t) e^{-xt^\alpha} dt,$$

where $a > 0$, $\alpha > 0$; $\varphi(t) = t^\beta(a_0 + a_1t + \dots)$, $\beta > -1$; $\max_{0 < t < a} |t^{-\beta}\varphi(t)| \leq M$; the series $\sum a_k t^k$ converges for $|t| \leq 2h$, $h > 0$. Then as $x \rightarrow \infty$

$$F(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{\alpha} \Gamma\left(\frac{\beta+n+1}{\alpha}\right) x^{-\frac{\beta+n+1}{\alpha}}. \quad (1)$$

Proof. First

$$F(x) = \int_0^h \varphi(t) e^{-xt^\alpha} dt + O(e^{-xh^\alpha}).$$

Then

$$\begin{aligned} \int_0^h \varphi(t) e^{-xt^\alpha} dt &= \frac{x^{-\frac{1}{\alpha}}}{\alpha} \int_0^{xh^{\frac{1}{\alpha}}} \varphi\left\{\left(\frac{u}{x}\right)^{\frac{1}{\alpha}}\right\} u^{\frac{1}{\alpha}-1} e^{-u} du = \\ &= \frac{1}{\alpha} \sum_{k=0}^{n-1} a_k \int_0^{xh^{\frac{1}{\alpha}}} \frac{u^{\frac{\beta+k+1}{\alpha}-1}}{x^{\frac{\beta+k+1}{\alpha}}} e^{-u} du + O\left(x^{-\frac{\beta+n+1}{\alpha}}\right), \end{aligned}$$

since for $|t| \leq h$

$$t^{-\beta} \varphi(t) - \sum_{k=0}^{n-1} a_k t^k = O(t^n),$$

and

$$\int_0^{xh^{\frac{1}{\alpha}}} u^{\rho-1} e^{-u} du < \int_0^{\infty} u^{\rho-1} e^{-u} du = \Gamma(\rho).$$

Since in addition

$$\int_0^{xh^{\frac{1}{\alpha}}} u^{\rho-1} e^{-u} du = \Gamma(\rho) + O\left(e^{-\frac{xh^{\frac{1}{\alpha}}}{2}}\right),$$

and $O(e^{-\epsilon x}) = O(x^{-n})$ for each fixed n and ϵ , we can write:

$$\int_0^h \varphi(t) e^{-xt^\alpha} dt = \frac{1}{\alpha} \sum_{k=0}^{n-1} \Gamma\left(\frac{\beta+k+1}{\alpha}\right) \frac{a_k}{x^{\frac{\beta+k+1}{\alpha}}} + O\left(x^{-\frac{\beta+n+1}{\alpha}}\right),$$

from which it follows that

$$F(x) \sim \frac{1}{\alpha} \sum_{k=0}^{\infty} \Gamma\left(\frac{\beta+k+1}{\alpha}\right) \frac{a_k}{x^{\frac{\beta+k+1}{\alpha}}}.$$

It is easy to see that this series diverges, if the coefficients a_k do not tend to zero too rapidly.

By dividing the interval (a, b) into parts and changing variables the same method can be applied to find an asymptotic series for the function

$$F(x) = \int_a^b \varphi(t) e^{xh(t)} dt.$$

We shall not write out this series in the general case, but shall merely indicate two of the most important special cases.

THEOREM 2. *Let $\varphi(t)$ and $h(t)$ be analytic functions of t , regular at each interior point of the interval $[a, b]$ and continuous on the closed interval $[a, b]$. Suppose further that $h(t)$ is real on $[a, b]$ and attains its largest value at the point $t = t_0$, $a < t_0 < b$, while $h''(t_0) < 0$.*

Then as $x \rightarrow +\infty$

$$\int_a^b \varphi(t) e^{xh(t)} dt \sim e^{xh(t_0)} \sqrt{\frac{\pi}{x}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!} \frac{a_{2n}}{x^n}, \quad (2)$$

where

$$\sum_{n=0}^{\infty} a_n t^n = \varphi(\psi(t)) \psi'(t),$$

$$\psi(\sqrt{h(t_0) - h(t)}) = t, \quad \psi(0) = t_0.$$

THEOREM 3. *If under the conditions of Theorem 2 $h(t)$ attains its largest value at the point $t=a$, and if $h(t)$ and $\varphi(t)$ are regular at $t=a$ with $h'(a) \neq 0$, then as $x \rightarrow \infty$*

$$\int_a^b \varphi(t) e^{xh(t)} dt \sim e^{xh(a)} \sum_{n=0}^{\infty} \frac{n! a_n}{x^{n+1}}, \quad (3)$$

where

$$\sum_{n=0}^{\infty} a_n t^n = \varphi(\psi(t)) \psi'(t),$$

$$\psi(h(a) - h(t)) = t, \quad \psi(0) = a.$$

The proof of each theorem consists in replacing the integral over $[a, b]$ by an integral over a sufficiently small neighborhood of the maximum point, after which the substitution $u = [h(t_0) - h(t)]^{1/2}$ (in the first case) or $u = h(a) - h(t)$ (in the second case) reduces the problem to Theorem 1.

It is easy to carry these results over to a more general case, and to make a large number of modifications.

We give two theorems relating to the general case.

Let

$$F(x) = \int_a^b \varphi(t) e^{h(x, t)} dt.$$

In what follows we shall assume that the following conditions are satisfied:

1. $h(x, t)$ and its first two derivatives in t are continuous on (a, b) .

2. For each fixed x , $h(x, t)$ attains its maximum value at just one point $t = t_0(x)$, $a < t_0(x) < b$, while $h''_t(x, t_0(x)) \rightarrow \infty$ for $x \rightarrow \infty$.

3. For each $\epsilon > 0$ there is an $x_1(\epsilon)$ such that for $x > x_1(\epsilon)$ the set of points t that satisfy the inequality

$$|\varphi(t)| e^{h(x,t)} > \frac{|\varphi(t_0(x))| e^{h(x,t_0(x))}}{q(x)^{1+\epsilon}}$$

($q(x) = [-h''(x, t_0(x))]^{1/2}$), forms an interval lying entirely inside the interval $[a, b]$. [Conditions 1, 2, 3 merely mean that the integrand has one sufficiently sharp maximum in (a, b) .]

If conditions 1, 2, 3 are satisfied then one has the following criterion for the application of the Laplace method.

THEOREM 4. *If for some $\epsilon > 0$ on the interval I_ϵ determined by*

$$|t - t_0(x)| < \frac{\sqrt{2(1+\epsilon) \ln^+ q(x)}}{q(x)}$$

$$\left(\text{here } \ln^+ A = \frac{\ln A + |\ln A|}{2} \right),$$

one has

$$\lim_{x \rightarrow \infty} \frac{\varphi(t)}{\varphi(t_0(x))} = 1, \quad \lim_{x \rightarrow \infty} \frac{h_t''(x, t)}{h_t''(x, t_0(x))} = 1$$

uniformly for $t \in I_\epsilon$, then

$$F(x) \sim \varphi(t_0(x)) e^{h(x, t_0(x))} \sqrt{-\frac{2\pi}{h_t''(x, t_0(x))}}. \quad (4)$$

Proof: We first show that for sufficiently large x the interval I_ϵ lies inside (a, b) . To this end we consider the integrand on the interval I_ϵ . We have:

$$\varphi(t) \sim \varphi(t_0(x)),$$

$$h(x, t) = h(x, t_0(x)) + \frac{(t - t_0(x))^2}{2!} h_t''(x, t_1),$$

where $t_1 = t_0(x) + \Theta(t - t_0(x))$, $0 < \Theta < 1$.

Therefore

$$h_t''(x, t_1) \sim h_t''(x, t_0(x)) = -q^2(x),$$

$$h(x, t) - h(x, t_0(x)) \sim -\frac{1}{2}(t - t_0(x))^2 q^2(x).$$

In particular at the endpoints of the interval I_*

$$\varphi(t) \sim \varphi(t_0(x)),$$

$$h(x, t) - h(x, t_0(x)) \sim -(1 + \varepsilon) \ln q(x),$$

and so

$$\frac{\varphi(t_0(x)) e^{h(x, t_0(x))}}{[q(x)]^{1+\frac{\varepsilon}{2}}} > |\varphi(t)| e^{h(x, t)} > \frac{\varphi(t_0(x)) e^{h(x, t_0(x))}}{[q(x)]^{1+2\varepsilon}} \quad (5)$$

for $x > x_0(\varepsilon)$.

By condition 3 the endpoints of the interval I_* , and therefore the whole interval, lie inside (a, b) .

On the other hand, on I_* for all x greater than some value the inequality

$$|\varphi(t)| e^{h(x, t)} < \frac{\varphi(t_0(x)) e^{h(x, t_0(x))}}{[q(x)]^{1+\frac{\varepsilon}{2}}}$$

must be satisfied, for otherwise inequality (5) would be violated.

Therefore

$$\begin{aligned} F(x) &= \int_a^b \varphi(t) e^{h(x, t)} dt = \\ &= \int_{I_*} \varphi(t) e^{h(x, t)} dt + \Theta M \frac{\varphi(t_0) e^{h(x, t_0)}}{[q(x)]^{1+\frac{\varepsilon}{2}}}, \end{aligned} \quad (6)$$

where $|\Theta| \leq 1$, and M is a constant not depending on x . But

$$\int_{I_0} \varphi(t) e^{h(x,t)} dt \sim \varphi(t_0) e^{h(x,t_0)} \int_{I_0} e^{-\frac{(t-t_0)^2}{2} q^2(x)} dx =$$

$$= \frac{\varphi(t_0) e^{h(x,t_0)}}{q(x)} \int_{-\frac{1}{2(1+\epsilon)\ln q(x)}}^{\frac{1}{2(1+\epsilon)\ln q(x)}} e^{-\frac{t^2}{2}} dt \sim \frac{\sqrt{2\pi} \varphi(t_0) e^{h(x,t_0)}}{q(x)}$$

Taking account of (6) we obtain the assertion of the theorem.

Theorem 4 is deficient in that it does not give an estimate of the remainder term, nor does it lead to the construction of an asymptotic series. The following theorem does not have these defects, but in other respects is somewhat less convenient.

THEOREM 5. *Suppose that conditions 1, 2, 3 are satisfied. Let the expression $h(x, t_0(x)) - h(x, t)$ increases as t approaches the point $t = t_0(x)$ in a sufficiently small neighborhood of $t_0(x)$. Let*

$$\alpha^2(x) = \min [h(x, t_0(x)) - h(x, t)]$$

(where the minimum is taken on t for t lying outside the intervals $(a_1, t_0(x))$, $(t_0(x), b_1)$, on which $h(x, t)$ is monotone). By $\psi(x, u)$ we denote the function defined in a neighborhood of $u=0$ by the conditions

$$h(x, t_0(x)) = h(x, \psi(x, u)) + u^2, \quad \psi(x, 0) = t_0(x).$$

Then for $x \rightarrow \infty$

$$F(x) = e^{h(x, t_0(x))} \sqrt{-\frac{2}{h''(x, t_0(x))}} \{S(x) + \varepsilon(x)\}, \quad (7)$$

where

$$\varepsilon(x) = O \left\{ \max_{t \in [t_1, t_2]} |\varphi(t)| e^{h(x, t) - h(x, t_0(x))} \sqrt{-h_t''(x, t_0(x))} \right\},$$

$$S(x) = \int_{-\alpha(x)}^{\alpha(x)} \varphi(\psi(x, u)) \frac{\psi'_u(x, u)}{\psi'_u(x, 0)} e^{-u^2} du.$$

Here $t_1 = t_1(x)$ and $t_2 = t_2(x)$ are points in the intervals of monotonicity of $h(x, t)$, adjacent to $t_0(x)$, defined by the conditions $h(x, t_1(x)) = h(x, t_2(x)) = h(x, t_0(x)) - [\alpha(x)]^2$.

Proof. By the choice of the points $t_1(x)$ and $t_2(x)$ and the definition of $\varepsilon(x)$

$$F(x) = \int_{t_1(x)}^{t_2(x)} \varphi(t) e^{h(x, t)} dt + O \left(\frac{\varepsilon(x) e^{h(x, t_0(x))}}{\sqrt{-h_t''(x, t_0(x))}} \right).$$

We make the change of variables

$$u = \sqrt{h(x, t_0(x)) - h(x, t)}$$

(the root is taken negative for $t < t_0(x)$ and positive for $t > t_0(x)$). By our notations $t = \psi(x, u)$. Since $h(x, t_1(x)) = h(x, t_0(x)) - [\alpha(x)]^2$, and $h(x, t_2(x)) = h(x, t_0(x)) - [\alpha(x)]^2$, the limits of integration will be $-\alpha(x)$, $\alpha(x)$, that is

$$F(x) = e^{h(x, t_0(x))} \left\{ \int_{-\alpha(x)}^{\alpha(x)} \varphi(\psi(x, u)) \psi'(x, u) e^{-u^2} du + O \left(\frac{\varepsilon(x)}{\sqrt{-h''(x, t_0(x))}} \right) \right\}$$

Since $\psi'_u(x, 0) = [-2/h''(x, t_0(x))]^{1/2}$, we obtain (7).

Clearly if $\alpha(x) \rightarrow \infty$ (sufficiently rapidly) and if

$$\frac{\psi'_u(x, u)}{\psi'_u(x, 0)} \rightarrow 1, \quad \frac{\varphi(\psi(x, u))}{\varphi(\psi(x, 0))} \rightarrow 1$$

for $|u| \leq \alpha(x)$, then we obtain the same results as in the preceding theorem. However, our theorem does not give sufficient conditions for this to happen.

On the other hand, if it can be shown that

$$\varphi(\psi(x, u)) \frac{\psi'_u(x, u)}{\psi'_u(x, 0)}$$

admits a suitable expansion in functions of x , then we can obtain an asymptotic series.

It should be noted that the theorems proven above remain true in most cases for the infinite interval. For the infinite interval the point where the maximum is attained may, in general, move out to infinity. However, by making a suitable linear substitution the maximum can be moved to a fixed point, and we obtain the same picture as for the finite interval with this difference that now we must also estimate the inessential part of the integral in a neighborhood of infinity.

Let us consider several examples where the Laplace method is applied.

$$1. \int_0^x e^{-t^2} dt = \Phi(x).$$

We have:

$$\Phi(x) = \frac{1}{2} \sqrt{\pi} - \int_x^\infty e^{-t^2} dt = \frac{1}{2} \sqrt{\pi} - e^{-x^2} \int_x^\infty e^{x^2-t^2} dt.$$

But

$$\int_x^\infty e^{x^2-t^2} dt = \frac{x}{2} \int_0^\infty \frac{e^{-ux^2}}{\sqrt{1+u}} du.$$

Hence by Theorem 1 we obtain:

$$\int_x^\infty e^{x^2-t^2} dt \sim \sum_{n=0}^\infty \frac{(2n)!}{2^{2n+1} n!} \frac{(-1)^n}{x^{2n+1}};$$

$$\Phi(x) \sim \frac{1}{2} \sqrt{\pi} - e^{-x^2} \sum_{n=0}^\infty \frac{(2n)!}{2^{2n+1} n!} \frac{(-1)^n}{x^{2n+1}}.$$

$$2. \quad \Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt.$$

Since $\Gamma(n+1) = n!$, Theorem 5 gives us the leading terms in Stirling's formula and determines the constant. With the notations of Theorem 5

$$h(x, t) = x \ln t - t, \\ t_0(x) = x, \quad h_t''(x, t_0(x)) = -\frac{1}{x}, \quad \alpha(x) = \infty,$$

and the function $\psi(x, u)$ is defined by the equations

$$x \ln x - x = x \ln \psi(x, u) - \psi(x, u) + u^2; \quad \psi(x, 0) = x.$$

From this we obtain

$$x \left\{ -\ln \frac{\psi(x, u)}{x} - 1 + \frac{\psi(x, u)}{x} \right\} = u^2$$

or, putting $\nu(z) = \ln \nu(z) - 1 = z^2$, $\nu(0) = 1$, then

$$\psi(x, u) = x \nu \left(\frac{u}{\sqrt{x}} \right), \\ \psi_u'(x, u) = \sqrt{x} \nu' \left(\frac{u}{\sqrt{x}} \right), \quad \frac{\psi_u'(x, u)}{\psi_u'(x, 0)} = \frac{\nu' \left(\frac{u}{\sqrt{x}} \right)}{\nu'(0)}.$$

Consequently,

$$\varepsilon(x) = 0, \quad S(x) = \int_{-\infty}^{\infty} \frac{\nu' \left(\frac{u}{\sqrt{x}} \right)}{\nu'(0)} e^{-u^2} du,$$

and we obtain for $\Gamma(x+1)$ the asymptotic series

$$\Gamma(x+1) \sim \sqrt{2\pi x} e^{x \ln x - x} \sum_{n=0}^{\infty} \frac{a_{2n} (2n)!}{x^n 2^{2n} n!}, \quad \frac{\nu'(u)}{\nu'(0)} = \sum_{n=0}^{\infty} a_n u^n,$$

since upon integration from $-\infty$ to ∞ the odd powers in the expansion of $\nu'(u)/\nu'(0)$ disappear.

$$3. F(x) = \int_0^{\infty} e^{tx-t \ln t} dt.$$

Here, with the notations of Theorem 5

$$h(x, t) = xt - t \ln t,$$

$$t_0(x) = e^{x-1}, \quad \alpha(x) = \infty, \quad h_t''(x, t_0(x)) = -e^{-x+1},$$

and the function $\psi(x, u)$ is defined by

$$e^{x-1} = x\psi(x, u) - \psi(x, u) \ln \psi(x, u) + u^2, \quad \psi(x, 0) = e^{x-1}.$$

This equation can be transformed into the form

$$\psi(x, u) e^{-x} \ln [\psi(x, u) e^{-x}] = u^2 e^{-x} - e^{-1},$$

from which it is clear that

$$\psi(x, u) e^{-x} = \nu\left(ue^{-\frac{x}{2}}\right),$$

where $\nu(t)$ satisfies the equations

$$\nu(t) \ln \nu(t) = t^2 - e^{-1} \text{ and } \nu(0) = e^{-1}.$$

Consequently,

$$\frac{\psi'_u(x, u)}{\psi'_u(x, 0)} = \frac{\nu'\left(ue^{-\frac{x}{2}}\right)}{\nu'(0)}, \quad s(x) = 0, \quad S(x) = \int_{-\infty}^{\infty} \frac{\nu'\left(ue^{-\frac{x}{2}}\right)}{\nu'(0)} e^{-u^2} du.$$

This gives us the estimate

$$F(x) = \sqrt{2\pi e^{x-1}} e^{e^{x-1}} (1 + O(e^{-x}))$$

and, if required, the asymptotic series

$$F(x) \sim \sqrt{2\pi e^{x-1}} e^{e^{x-1}} \sum_{n=0}^{\infty} \frac{a_{2n}(2n)!}{2^{2n} n!} e^{-nx},$$

where the coefficients a_n are determined from the equations

$$\sum_{n=0}^{\infty} a_n u^n = \frac{v'(u)}{v'(0)}, \quad v(u) \ln v(u) = u^2 - \frac{1}{e}, \quad v(0) = \frac{1}{e}.$$

The possibilities of applying Laplace's method in analysis are very great. Especially valuable is a modification of this method, applied to contour integrals of analytic functions. This is the so-called saddlepoint method (see section 5).

4. The method of generating functions

At first glance it might seem that the theory of analytic functions is related to asymptotic estimates only in so far as most of the functions encountered in analysis are analytic. However the role of the theory of analytic functions in obtaining asymptotic estimates is much broader. In obtaining asymptotic estimates of any quantity it is almost always helpful to use the powerful apparatus of the theory of analytic functions. The idea of using this apparatus consists in applying some transformation to replace the given function by an analytic function.

At first we shall only consider examples involving the simplest variant of this method. This simplest variant consists of the following.

Suppose we wish to investigate the asymptotic behaviour of a sequence $\{a_n\}$, $n=0, 1, 2, \dots$. We consider the analytic function

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The properties of the sequence $\{a_n\}$ may enable us to determine the singularities of the function $F(z)$. [We

assume that $F(z)$ has singular points in the finite plane.] The function $F(z)$, and therefore also the sequence $\{a_n\}$, is completely determined by all its singular points. However, not all singularities play an equal role in the growth of $\{a_n\}$. The singularities that are closer to $z=0$ have greater influence, the more distant ones have less influence. The influence of singular points lying at the same distance is also varied, those singularities near which $F(z)$ grows most rapidly have the greatest influence on the growth of $\{a_n\}$. If we separate the "principal" singular points, we find that the coefficients of $F_1(z)$ give the leading term in the asymptotic behaviour of the sequence $\{a_n\}$ as $n \rightarrow \infty$. At the same time the function with fewer singularities is simpler, and therefore has simpler coefficients. Thus one can expect that this method will enable us to obtain good asymptotic estimates for $\{a_n\}$.

Let us consider some examples.

1. The function $a_n(x)$ is defined by the recurrence relation

$$a_{n+1}(x) = \frac{a_n(x)}{1!} + \frac{a_{n-1}(x)}{2!} + \dots + \frac{a_0(x)}{(n+1)!} + \frac{x^{n+1}}{(n+1)!}, \quad (1)$$

$$a_0(x) = 1.$$

We wish to study the asymptotic behaviour of $a_n(x)$ for fixed x as $n \rightarrow \infty$.

Let

$$F(z, x) = \sum_{n=0}^{\infty} a_n(x) z^n.$$

Multiply both sides of (1) by z^{n+1} and sum from $n = -1$ to ∞ . We obtain:

$$\begin{aligned}
F(z, x) &= 1 + \frac{xz}{1!} + \frac{x^2 z^2}{2!} + \dots + a_0(x) \frac{z}{1!} + a_0(x) \frac{z^2}{2!} + \dots \\
&\dots + a_1(x) \frac{z}{1!} z + a_1(x) \frac{z^2}{2!} z + \dots = e^{xz} + a_0(x)(e^z - 1) + \\
&\quad + a_1(x) z(e^z - 1) + \dots = e^{xz} + (e^z - 1)F(z, x).
\end{aligned}$$

From this equation we find:

$$F(z, x) = \frac{e^{xz}}{2 - e^z}.$$

The only singular points of $F(z, x)$ are the zeros of the denominator, that is, simple poles at the points $z_k = \ln 2 + 2k\pi i$. The nearest singular point to $z=0$ is the point $z_0 = \ln 2$, a simple pole with the residue -2^{x-1} . The simplest function having this singularity is

$$F_1(z, x) = \frac{2^{x-1}}{\ln 2 - z}.$$

The function $F(z, x) - F_1(z, x)$ has no singularities for $|z| < (4\pi^2 + \ln^2 2)^{1/2}$, and therefore for the coefficients of this difference we have the estimate

$$\varepsilon_n(x) = o((2\pi)^{-n}), \quad F(z, x) - F_1(z, x) = \sum_{n=0}^{\infty} \varepsilon_n(x) z^n.$$

Consequently,

$$a_n(x) = 2^{x-1} (\ln 2)^{-n-1} + \varepsilon_n(x) = 2^{x-1} (\ln 2)^{-n-1} + o((2\pi)^{-n}).$$

If desired one can obtain for $a_n(x)$ the asymptotic series

$$\begin{aligned}
&a_n(x) \sim \\
&\sim 2^{x-1} \left\{ (\ln 2)^{-n-1} + \sum_{k=1}^{\infty} \frac{e^{2k\pi i x}}{(\ln 2 + 2k\pi i)^{n+1}} + \frac{e^{-2k\pi i x}}{(\ln 2 - 2k\pi i)^{n+1}} \right\}.
\end{aligned}$$

2. Let the function $P_n(x)$ be defined by the equation

$$P_n(x) = \int_0^x \int_1^{x_1} \dots \int_{\frac{1}{2} + \frac{1}{2}(-1)^n}^{x_{n-1}} dx_1 \dots dx_n, \quad P_0(x) = 1. \quad (2)$$

Let us find the asymptotic behaviour of $P_n(x)$ for x fixed as $n \rightarrow \infty$.

For $F(z, x) = \sum z^n P_n(x)$ we obtain the differential equation

$$\frac{\partial^2}{\partial x^2} F(z, x) = z^2 F(z, x); \quad F(z, 0) = 1, \quad \frac{\partial}{\partial x} F(z, 1) = z,$$

since

$$P_n''(x) = P_{n-2}(x); \quad P_n(0) = 0, n \geq 1; \quad P_n'(1) = 0, n \geq 2.$$

Solving this equation we obtain:

$$F(z, x) = C_1(z) e^{xz} + C_2(z) e^{-xz};$$

$$C_1(z) + C_2(z) = 1, \quad C_1(z) e^z - C_2(z) e^{-z} = 1,$$

from which

$$F(z, x) = \frac{\operatorname{ch}(x-1)z + \operatorname{sh} xz}{\operatorname{ch} z}.$$

The nearest singularities of $F(z, x)$ to $z=0$ are the points $z = \pm \pi i/2$, consequently, the principal term in the asymptotic expansion of $P_n(x)$ is given by the coefficients of the function

$$\begin{aligned} F_1(z, x) &= \frac{\operatorname{ch}(x-1)\frac{\pi l}{2} + \operatorname{sh}\frac{\pi l x}{2}}{\left(z - \frac{\pi l}{2}\right) \operatorname{sh}\frac{\pi l}{2}} - \frac{\operatorname{ch}(x-1)\frac{\pi l}{2} - \operatorname{sh}\frac{\pi l x}{2}}{\left(z + \frac{\pi l}{2}\right) \operatorname{sh}\frac{\pi l}{2}} = \\ &= \frac{\pi + 2z}{z^2 + \frac{\pi^2}{4}} \sin \frac{\pi x}{2}. \end{aligned}$$

Since the difference $F(z, x) - F_1(z, x)$ has no singularities for $|z| < 3\pi/2$, we obtain for $P_n(x)$ the asymptotic estimate

$$P_n(x) = 2 \left(\frac{2}{\pi}\right)^{n+1} (-1)^{\left[\frac{n}{2}\right]} \sin \frac{\pi x}{2} + O\left(\left(\frac{2}{3\pi}\right)^n\right).$$

By considering more distant points, one could again obtain an asymptotic series.

In our examples we limited ourselves to cases in which the generating function had only simple poles. This was not essential for the method—we could also admit singular points of the form $(z-z_k)^\alpha \varphi(z)$, where $\varphi(z)$ is regular for $z=z_k$, and even somewhat more complicated singularities. However, the supply of singularities for which we know the asymptotics of the coefficients of functions having these singularities, is not very large. Thus the method of generating functions can be successfully applied to obtain asymptotic estimates only in those cases where we have a good method of obtaining asymptotic estimates for the coefficients of the auxiliary function. Such a method is the saddlepoint method, of which we shall speak in the following section.

Here we only note that in the method of generating functions the particular construction we have used up till now is not essential. In other cases it may be more convenient to associate with the sequence $\{a_n\}$ the series

$$F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

or even the product

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right).$$

For functions of a continuous rather than a discrete variable one can take for the generating function, for example,

$$F(z) = \int_0^{\infty} a(x) e^{-zx} dx.$$

The function $F(z)$ is called the *Laplace transform of the function $a(x)$* .⁴

It is, of course, entirely possible to apply other integral transformations. For example, one very frequently uses the transformation

$$F(z) = \int_0^{\infty} x^{z-1} a(x) dx,$$

known as the *Mellin transform*.

Now let us consider some examples in which integral transforms are used to obtain asymptotic estimates. For this purpose we shall need a formula permitting us to find $a(x)$ from $F(z)$ —an analogue of Cauchy's formula for the coefficients of a power series.

If $a(x)$ is continuous for $0 < x < \infty$ and

$$\int_0^{\infty} |a(x)| e^{-\sigma x} dx < \infty,$$

then one has the formula

$$a(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} F(z) e^{xz} dz, \quad F(z) = \int_0^{\infty} a(t) e^{-zt} dt, \quad (3)$$

⁴This transform has other uses besides the obtaining of asymptotic estimates. It is the basis of the operational calculus.

known as the *inversion formula for the Laplace transform*. Here, and also in what follows, the first integral is taken along the vertical line of abscissa α and must be understood, in general, as a principal value.

To prove this formula it is sufficient to note that the integral for $F(z)$ converges uniformly for $\operatorname{Re} z \geq \alpha$. Then, multiplying by e^{xz} and integrating on z from $\alpha - iR$ to $\alpha + iR$, we obtain:

$$\begin{aligned} \int_{\alpha - iR}^{\alpha + iR} F(z) e^{xz} dz &= 2i \int_0^{\infty} a(t) e^{\alpha(x-t)} \frac{\sin(x-t)R}{x-t} dt = \\ &= 2ie^{\alpha x} \int_{-\infty}^{\infty} a(x+u) e^{-(x+u)} \frac{\sin uR}{u} du = \\ &= 2ie^{\alpha x} \int_{-\infty}^{\infty} b\left(x + \frac{u}{R}\right) \frac{\sin u}{u} du, \quad b(x) = a(x) e^{-\alpha x}. \end{aligned}$$

But for $R \rightarrow \infty$ the last integral tends to

$$b(x) \int_{-\infty}^{\infty} \frac{\sin u}{u} du = \pi b(x),$$

from which we obtain (3).

In the same way it can be shown that the *inversion formulae for the Mellin transform* are

$$a(x) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} F(z) x^{-z} dz, \quad F(z) = \int_0^{\infty} a(t) t^{z-1} dt. \quad (4)$$

We give two examples where asymptotic estimates are obtained with the aid of these integral transforms.

3. Find the asymptotic behaviour of the function

$$\varphi(x) = \sum_{n=1}^{\infty} e^{-xn^{\frac{1}{\alpha}}}, \quad \alpha > 0,$$

as $x \rightarrow 0$, $x > 0$.

We apply the Mellin transform. We have for $\operatorname{Re} z > \alpha$.

$$\begin{aligned} F(z) &= \int_0^{\infty} x^{z-1} \varphi(x) dx = \sum_{n=1}^{\infty} \int_0^{\infty} x^{z-1} e^{-xn^{\frac{1}{\alpha}}} dx = \\ &= \sum_{n=1}^{\infty} n^{-\frac{z}{\alpha}} \int_0^{\infty} t^{z-1} e^{-t} dt = \Gamma(z) \zeta\left(\frac{z}{\alpha}\right), \end{aligned}$$

where $\zeta(z) = \sum n^{-z}$ is the Riemann zeta function. We investigate the behaviour of $\zeta(z)$ for large $|y|$ ($z = \sigma + iy$). For $\operatorname{Re} z > 1$ we have:

$$\begin{aligned} \zeta(z) &= \int_0^{\infty} x^{-z} d[x] = z \int_0^{\infty} \frac{[x]}{x^{z+1}} dx = \\ &= \frac{z}{z-1} - \frac{1}{2} + z \int_0^{\infty} \frac{[x] - x + \frac{1}{2}}{x^{z+1}} dx. \end{aligned}$$

If we successively integrate by parts and introduce the functions $\psi_1(x)$, $\psi_2(x)$ as in example 4 Section 1, etc., we easily see that $\zeta(z)$ is regular for all z , except for $z=1$, and that for any σ

$$\zeta(\sigma + iy) = O(|y|^{k(\sigma)}).$$

With respect to $\Gamma(z)$, we shall see in the following section that

$$\ln \Gamma(z) \sim z \ln z - z, \quad |\arg z| \leq \pi - \eta,$$

from which it is clear that

$$\Gamma(\sigma + iy) = O\left(e^{-\frac{\pi}{2}|y|}\right)$$

for each fixed σ . This means that for any σ ,

$$\int_{-\infty}^{\infty} \left| \zeta\left(\frac{z+iy}{\alpha}\right) \Gamma(\sigma+iy) \right| dy < \infty.$$

Thus our generating function is absolutely integrable on each line parallel to the imaginary axis, on which there are no poles of the integrand. The singular points of $F(z)$ are the point $z=\alpha$ —the pole of $\zeta(z/\alpha)$, and the points $0, -1, -2, \dots$ —the poles of $\Gamma(z)$. These singular points are all simple poles. (The singular points of $\Gamma(z)$ are easy to find, if $\Gamma(z)$ is represented in the form

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt + \Gamma_1(z), \quad \Gamma_1(z) = \int_1^{\infty} t^{z-1} e^{-t} dt;$$

$\Gamma_1(z)$ is an entire function, and

$$\int_0^1 t^{z-1} e^{-t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 t^{n+z-1} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{n+z},$$

from which it is clear that the singular points of $\Gamma(z)$ are only simple poles at the points $z = -n, n=0, 1, 2, \dots$, with the residues $(-1)^n/n!$.)

Just as in the previous examples, where the singularities nearest the origin had the greatest influence, here the singularities with the largest real part have the greatest influence.

To obtain estimates we use the inversion formula (4). We have:

$$\varphi(x) = \frac{1}{2\pi i} \int_{\alpha+\eta-i\infty}^{\alpha+\eta+i\infty} \Gamma(z) \zeta\left(\frac{z}{\alpha}\right) x^{-z} dz, \quad \eta > 0.$$

Moving the contour of integration to the left of the line $\operatorname{Re} z = \alpha$, but to the right of the line $\operatorname{Re} z = 0$, we obtain by the residue theorem

$$\varphi(x) = x^{-\alpha} \Gamma(\alpha + 1) + \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} \Gamma(z) \zeta\left(\frac{z}{\alpha}\right) x^{-z} dz, \quad 0 < \eta < \alpha.$$

But

$$\left| \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} \Gamma(z) \zeta\left(\frac{z}{\alpha}\right) x^{-z} dz \right| \leqslant \\ \leqslant x^{-\eta} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \Gamma(\eta + iy) \zeta\left(\frac{\eta + iy}{\alpha}\right) \right| dy = O(x^{-\eta}),$$

so that

$$\varphi(x) = x^{-\alpha} \Gamma(\alpha + 1) + O(x^{-\eta}), \quad \eta > 0.$$

If we shift the contour to the left again, then in the same way we obtain

$$\varphi(x) = x^{-\alpha} \Gamma(\alpha + 1) + \sum_{n=0}^N \frac{(-1)^n}{n!} \zeta\left(-\frac{n}{\alpha}\right) x^n + \\ + O(x^{N+1-\eta}), \quad \eta > 0.$$

Thus we can write for $\varphi(x)$ the asymptotic series

$$\varphi(x) \sim x^{-\alpha} \Gamma(\alpha + 1) + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \zeta\left(-\frac{n}{\alpha}\right) x^n.$$

This formula completely solves the problem of the asymptotic behaviour of $\varphi(x)$ as $x \rightarrow \infty$. It is valid not only for $x \rightarrow 0$ through positive values, but for any manner of approach within the angle

$$|\arg x| \leq \frac{\pi}{2} - \eta, \quad \eta > 0.$$

We note that for $\alpha = 1/2k$ (k a positive integer) all terms in the asymptotic expansion vanish, but that this does not mean that $\varphi(x) = \Gamma(\alpha+1)/x^\alpha$. This merely means that the difference $\varphi(x) - \Gamma(\alpha+1)/x^\alpha$ decreases faster than any power of x as $x \rightarrow 0$.

4. Let us find an asymptotic estimate as $|z| \rightarrow \infty$ for the function

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\frac{n}{\rho} + 1\right)}.$$

We shall find $F(z)$, the Laplace transform of $E_\rho(x^{1/\rho})$. Since for $x > 0$

$$\begin{aligned} E_\rho\left(x^{\frac{1}{\rho}}\right) &\sim \int_0^\infty \frac{x^{\frac{t}{\rho}}}{\Gamma\left(\frac{t}{\rho} + 1\right)} dt = \\ &= \rho \int_0^\infty \frac{x^u}{\Gamma(u+1)} du \sim \rho \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} = \rho e^x. \end{aligned}$$

(these calculations, of course, are merely suggestive; we shall verify the correctness of the estimate below), it follows that for $\operatorname{Re} z > 1$ the integral for $F(z)$ converges, and we can integrate term-by-term. We have:

$$\begin{aligned} F(z) &= \int_0^\infty E_\rho\left(x^{\frac{1}{\rho}}\right) e^{-xz} dx = \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(\frac{n}{\rho} + 1\right)} \int_0^\infty x^{\frac{n}{\rho}} e^{-xz} dx = \\ &= \sum_{n=0}^{\infty} \frac{z^{-\frac{n}{\rho}-1}}{\Gamma\left(\frac{n}{\rho} + 1\right)} \int_0^\infty u^{\frac{n}{\rho}} e^{-u} du = \sum_{n=0}^{\infty} z^{-\frac{n}{\rho}-1} = \frac{z^{\frac{1}{\rho}-1}}{z^{\frac{1}{\rho}} - 1}. \end{aligned}$$

All operations are legitimate for $z > 1$, but by the principle of analytic continuation this means that

$$F(z) = \frac{z^{\frac{1}{p}-1}}{\frac{1}{z^p}-1}$$

for all z . The singularity of the generating function that is furthest to the right is a simple pole at $z=1$. This determines the principal part of the growth of $E_p(x^{1/p})$. The further asymptotics are determined by the point $z=0$. To obtain these asymptotics we must use the inversion formula, as in example 3. By the residue theorem we have

$$E_p\left(x^{\frac{1}{p}}\right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{z^{\frac{1}{p}-1} e^{xz}}{\frac{1}{z^p}-1} dz = \rho e^x + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{z^{\frac{1}{p}-1} e^{xz}}{\frac{1}{z^p}-1} dz.$$

Introducing the new variable of integration $u=xz$ and replacing $x^{1/p}$ by z , we obtain:

$$E_p(z) = \rho e^{z^p} + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{u^{\frac{1}{p}-1} e^u}{\frac{1}{u^p}-z} du.$$

The contour of integration in this formula can be arbitrarily changed, if the convergence is not disturbed and if we do not cross over any singular points. In particular the negative real axis described twice can be taken for the contour. Then

$$E_p(z) = \rho e^{z^p} + \frac{1}{2\pi i} \int_L \frac{u^{\frac{1}{p}-1} e^u}{\frac{1}{u^p}-z} du, \quad (5)$$

$$L(-\infty - 0i, \quad 0, \quad -\infty + 0i).$$

It is immediately clear from (5) that the integral tends to zero as $z \rightarrow \infty$, $z > 0$, that is, $E_\rho(z) = \rho e^{z^\rho} + o(1)$ for $z > 0$, which confirms the estimate that was derived in a nonrigorous manner at the beginning of the example.

One can continue and obtain an asymptotic series. Namely:

$$\begin{aligned} \frac{1}{2\pi i} \int_L \frac{u^{\frac{1}{\rho}-1} e^u}{u^{\frac{1}{\rho}} - z} du &\sim \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \quad a_n = -\frac{1}{2\pi i} \int_L u^{\frac{n}{\rho}-1} e^u du = \\ &= \frac{e^{-\frac{\pi n i}{\rho}} - e^{\frac{\pi n i}{\rho}}}{2\pi i} \int_0^{\infty} x^{\frac{n}{\rho}-1} e^{-x} dx = -\frac{1}{\pi} \sin \frac{\pi n}{\rho} \Gamma\left(\frac{n}{\rho}\right). \end{aligned}$$

Thus for $z > 0$ we finally obtain:

$$E_\rho(z) \sim \rho e^{z^\rho} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{\rho} \Gamma\left(\frac{n}{\rho}\right)}{z^n}. \quad (6)$$

But formula (5) is valid not only for positive z , but for all z for which

$$\min_{u \in L} \left| u^{\frac{1}{\rho}} - z \right| \geq c > 0,$$

since with this condition

$$\left| \int_L \frac{u^{\frac{n}{\rho}-1} e^u}{\left(u^{\frac{1}{\rho}} - z\right) z^n} du \right| \leq \frac{1}{|z|^n} \int_0^{\infty} \frac{x^{\frac{n}{\rho}-1} e^{-x}}{c_1 + \left|x^{\frac{1}{\rho}} - |z|\right|} dx = o\left(\frac{1}{|z|^n}\right).$$

For $\rho \leq \frac{1}{2}$ this condition means that z must lie outside the half-planes $\operatorname{Re} z < 0$, $|\operatorname{Im} z| < c$, $c > 0$. For each fixed x this gives us the uniform estimate in z

$$E_\rho(z) \sim \rho e^{z^\rho} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{\rho} \Gamma\left(\frac{n}{\rho}\right)}{z^n}. \quad (7)$$

For $\frac{1}{2} < \rho \leq 1$ this formula gives an estimate for all z .

If $\rho > 1$, then in (5) we can change the variable of integration and put $u^{1/\rho} = t$. Then the contour L is transformed into a contour L' , consisting of two rays

$$\left(\infty e^{-\frac{\pi i}{\rho}}, 0\right), \quad \left(0, \infty e^{\frac{\pi i}{\rho}}\right).$$

For $E_\rho(z)$ we obtain the formula

$$E_\rho(z) = \rho e^{z^\rho} + \frac{\rho}{2\pi i} \int_{L'} \frac{e^{t^\rho}}{t-z} dt, \quad |\arg z| < \frac{\pi}{\rho}. \quad (8)$$

For $\pi/\rho < |\arg z| \leq \pi$ we have by the residue theorem:

$$E_\rho(z) = \frac{\rho}{2\pi i} \int_{L'} \frac{e^{t^\rho}}{t-z} dt, \quad \frac{\pi}{\rho} < |\arg z| \leq \pi. \quad (9)$$

Since the asymptotic series for the integral remains the same, we obtain finally the estimate

$$\left. \begin{aligned} E_\rho(z) &\sim \rho e^{z^\rho} + \sum_{n=1}^{\infty} \frac{a_n}{z^n}, & |\arg z| &\leq \frac{\pi}{2\rho} + \eta, \\ E_\rho(z) &\sim \sum_{n=1}^{\infty} \frac{a_n}{z^n}, & \frac{\pi}{2\rho} + \eta &\leq |\arg z| \leq \pi. \end{aligned} \right\} \quad (10)$$

Here, as before,

$$a_n = -\frac{1}{\pi} \sin \frac{\pi n}{\rho} \Gamma\left(\frac{n}{\rho}\right).$$

Formula (10) is the most general and is suitable both for arbitrary $\rho > \frac{1}{2}$, and for $\rho \leq \frac{1}{2}$, with the exception of estimates close to the negative part of the real axis.

5. The saddlepoint method

The saddlepoint method consists, roughly speaking, in the application of the idea of Laplace's method for the asymptotic estimation of integrals of real functions, to contour integrals of the form

$$\int_C F(z, t) dt$$

for large values of the parameter z (not necessarily real). The class of quantities that can be represented by integrals of this sort is very large. In particular, the Taylor coefficients of an analytic function $F(z)$ are expressed by Cauchy's formula in the form

$$a_n = \frac{1}{2\pi i} \int_C \frac{F(t)}{t^{n+1}} dt,$$

which in most cases will satisfy the conditions that are needed to apply the saddlepoint method. Integrals of a similar type give the inversion formulae for the Laplace and Mellin transforms. Thus the saddlepoint method is an essential supplement to the method of generating functions. However the saddlepoint method has many other applications, unrelated to the method of generating functions.

We shall first show how to apply the Laplace method to the problem of estimating curvilinear integrals, and then we shall discuss the specific features of the saddlepoint method.

Suppose we are given the integral

$$\frac{1}{2\pi i} \int_C e^{h(z, t)} dt,$$

where $h(z, t)$ is an analytic function, regular in t in some domain D , and C is a contour which may or may not be closed, lying in this domain (D will not in general be simply connected). The essence of the saddlepoint method consists in using Cauchy's theorem on the independence of the path of integration to choose the contour C so that, in a neighborhood of the point where $|e^{h(z, t)}|$ attains its largest value on the contour C , $\arg e^{h(z, t)}$ is constant. If such a contour can be found, then what remains to be done is an ordinary estimation of an integral of a real function by the Laplace method. The corresponding formulae carry over to this case with no change. We shall write out these formulae later, but now we note a basic property of such contours.

THEOREM 1. *Let $F(t)$ be an analytic function, regular in a domain D . If the point t_0 in D is such that there is a contour C passing through it having the properties:*

- (1) *$\operatorname{Re} F(t)$ attains its maximum on C at the point t_0 ;*
- (2) *$\operatorname{Im} F(t)$ is constant on C ,*

then $F'(t_0) = 0$.

Proof: Since $\operatorname{Im} F(t) = \text{const}$ on C , C is an analytic curve. Let $t = t(s)$ be the equation for C , where s is arc length along C , measured from $t = t_0$. Then from (2) we have $(d/ds)\operatorname{Im} F(t(s)) = 0$, and from (1) we have

$$\frac{d}{ds} \operatorname{Re} F(t(s)) \Big|_{s=0} = 0.$$

Therefore

$$\frac{d}{ds} F(t(s)) \Big|_{s=0} = F'(t_0) t'(0) = 0.$$

but $t'(0) \neq 0$, which means that $F'(t_0) = 0$, and the theorem is proven.

The points where $F'(t_0) = 0$ are called *saddlepoints*, and the curves $\operatorname{Im} F(t) = \text{const}$ are called *paths of steepest*

descent. This is the origin of the name of the method. The meaning of these names becomes clear if we consider the geometric surface $u = u(x, y) = \operatorname{Re} F(x + iy)$. Indeed, $u = u(x, y)$ is a harmonic function. Since harmonic functions cannot attain a maximum or a minimum inside the domain of harmonicity, it follows that the surface $u = \operatorname{Re} F(t)$ must be saddle-shaped at the point $t = t_0$. Also, if n is any direction then

$$\left(\frac{\partial \operatorname{Re} F}{\partial n}\right)^2 + \left(\frac{\partial \operatorname{Im} F}{\partial n}\right)^2 = |F'(t)|^2.$$

This means that $\partial u / \partial \bar{n}$ will be largest in the direction in which $(\partial / \partial \bar{n}) \operatorname{Im} F = 0$, that is, in the direction tangent to the curve $\operatorname{Im} F(t) = \text{const}$. This means that the direction of the curve $\operatorname{Im} F(t) = \text{const}$ is the direction of greatest slope of the surface $u = \operatorname{Re} F(t)$, that is, the direction of steepest descent. (The saddlepoint method is sometimes called the method of steepest descent.)

Thus the recipe for applying Laplace's method to the estimation of contour integrals consists in passing the contour through a saddlepoint of the surface $\operatorname{Re} h(z, t)$, while in a neighborhood of the saddlepoint the contour is taken in the direction of steepest descent. In this manner an integral is obtained that can be estimated by Laplace's method.

If the contour is not closed it may happen that the only possibility of obtaining a contour on which $\operatorname{Im} h(z, t)$ is constant and such that $\operatorname{Re} h(z, t)$ attains its largest value on C at a certain point is when this point is an endpoint of the contour. In such a case one still chooses for the contour a path of steepest descent passing through that endpoint, and thus reduces the problem of estimation to the Laplace method.

Up till now we have not considered the questions of whether it is always possible to pass the contour C through a saddlepoint in such a way that the largest value of $\operatorname{Re} h(z, t)$ on the contour is attained at the saddlepoint, or of how to choose the saddlepoint (if there are several such points), through which the contour is to be passed. The fact is that the answers to these questions are not at all simple, and that satisfactory general criteria are not known (for the second question). In Chapter III in the estimation of certain special classes of entire functions we present certain considerations which can be applied, it seems, only to these classes; here we merely note that these and similar questions that are very complicated in the general case can often be solved very simply in concrete situations.

Now we derive the formulae that enable us to write down asymptotic estimates, if we already know through which saddlepoint the contour can be passed. (There may be only one such possibility.)

First of all assume that the integral has the form

$$F(z) = \frac{1}{2\pi i} \int_C \varphi(t) e^{zh(t)} dt,$$

and that the contour C can be made to pass through the saddlepoint $t=t_0$, where the value $\operatorname{Re} h(t_0)$ exceeds the value of $\operatorname{Re} h(t)$ at all other saddlepoints on the contour. Suppose further that $h''(t_0) \neq 0$ and that t_0 is not an endpoint of the contour. Then the Laplace method can be applied to estimate $F(z)$ and by Theorem 2, Section 3 we obtain:

$$F(z) \sim \pm \frac{e^{zh(t_0)}}{\sqrt{-i\pi z}} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!} \frac{a_{2n}}{z^n} i, \quad (1)$$

where the coefficients a_n are determined from the formulae

$$\sum_{n=0}^{\infty} a_n u^n = \varphi(\psi(u)) \psi'(u),$$

$$h(\psi(u)) = h(t_0) - u^2, \quad \psi(0) = t_0.$$

If there are two or more saddlepoints on the contour at which the value of $\operatorname{Re} h(t)$ is close to its maximum, then it is necessary to take the sum over all these points. The sign is determined by the direction of the contour.

If the contour is not closed and the largest value is taken at an endpoint, then the asymptotic estimate of $F(z)$ is given by Theorem 3, Section 3:

$$F(z) \sim \pm \frac{\varphi(t_0) e^{zh(t_0)}}{2\pi i} \sum_{n=0}^{\infty} \frac{a_n n!}{z^n}, \quad (2)$$

where the coefficients a_n are determined by the formulae

$$\sum_{n=0}^{\infty} a_n u^n = \varphi(\psi(u)) \psi'(u),$$

$$h(\psi(u)) - u = h(t_0), \quad \psi(0) = t_0.$$

Now let

$$F(z) = \frac{1}{2\pi i} \int_C \varphi(t) e^{h(z,t)} dt$$

and assume that the contour C can be made to pass through the saddle point $t=t_0(z)$ with the largest value, that t_0 is not an endpoint of the contour, and that $h''(z, t_0(z)) \neq 0$.

Suppose also that for each fixed $\epsilon > 0$ we have, uniformly in z in the circle

$$|t - t_0(z)| \leq r_\epsilon(z),$$

where

$$r_*(z) = \sqrt{\frac{2(1+\varepsilon)\ln|h_t''(z, t_0(z))|}{|h_t''(z, t_0(z))|}},$$

$$\lim_{z \rightarrow \infty} \frac{\varphi(t)}{\varphi(t_0(z))} = 1, \quad \lim_{z \rightarrow \infty} \frac{h''(z, t)}{h''(z, t_0(z))} = 1,$$

and that the endpoints of the contour do not lie in this circle. Then from Theorem 4, Section 3 we obtain for $F(z)$ the estimate

$$F(z) \sim \pm \frac{\varphi(t_0(z)) e^{h(z, t_0(z))}}{\sqrt{2\pi h_t''(z, t_0(z))}} \quad (3)$$

(the sign again depends on the direction of the contour).

Finally, if conditions corresponding to the conditions of Theorem 5, Section 3 are satisfied we have the formula

$$F(z) \sim \pm \frac{e^{h(z, t_0(z))}}{2\pi} \sqrt{\frac{2}{h''(z, t_0(z))}} \{S(z) + \varepsilon(z)\}, \quad (4)$$

$$S(z) = \int_{-\alpha(z)}^{\alpha(z)} \varphi(\psi(z, u)) \frac{\psi'_u(z, u)}{\psi'_u(z, 0)} e^{-u^2} du,$$

$$|\varepsilon(z)| = O \left\{ \sqrt{h_t''(z, t_0(z))} e^{-\alpha(z)} \max_{|t-t_0| \leq \alpha(z)} |\varphi(t)| \right\},$$

where $\Psi(zu)$ is determined by the equation

$$h(z, t_0(z)) = h(z, \psi(z, u)) + u^2, \quad \psi(z, 0) = t_0(z),$$

and $\alpha(z)$ is determined just as in Theorem 5, Section 3. (In most cases $\alpha(z) = \infty$.)

In general, each variant of Laplace's method leads to analogous formulae in the saddlepoint method.

Let us consider some examples of the saddlepoint method.

$$1. \quad \Gamma(z+1) = \int_0^{\infty} t^z e^{-t} dt.$$

We have already investigated the function $\Gamma(x+1)$ for real values of the argument and we obtained Stirling's formula in this case. By applying the saddlepoint method instead of Laplace's method we can obtain this same formula for all z satisfying $|\arg z| \leq \pi - \eta$, $\eta > 0$. Indeed, there is only the one saddlepoint $t_0(z) = z$, the value of $\operatorname{Re} h(z, t)$ for $t=0$ and $t = \infty$ is $-\infty$, so that the asymptotics of $\Gamma(z+1)$ are just given by the saddlepoint. The estimates at this point are the same and so

$$\Gamma(z+1) = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right).$$

(At first glance it may seem that for $|\arg z| > \pi/2$ there is no chance for an estimate because of the divergence of the integral, however this difficulty is easily overcome by shifting the contour.)

$$2. \quad \Gamma_1(z+1) = \int_1^{\infty} t^z e^{-t} dt.$$

In this case the situation is complicated since in addition to the saddlepoint $t_0(z) = z$ we must also consider the point $t=1$, the endpoint of the contour. Therefore

$$\Gamma_1(z+1) = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right) - \frac{1}{ze} \left(1 + O\left(\frac{1}{z}\right)\right). \quad (5)$$

It is easy to see that for $|\arg z| \leq \frac{1}{2}\pi - \eta$ the first term plays the leading role, while for $\frac{1}{2}\pi + \eta \leq |\arg z| \leq \pi$, it is the second term. In the remaining interval the two terms are more or less equal.

3. We shall obtain asymptotic estimates of the Bessel functions by using their generating function

$$e^{\frac{\sigma}{2}(s-\frac{1}{s})} = \sum_{n=-\infty}^{\infty} z^n J_n(x). \quad (6)$$

From (6) we obtain

$$J_n(x) = \frac{1}{2\pi i} \int_{|z|=1} e^{\frac{\sigma}{2}(t-\frac{1}{t})} \frac{dz}{t^{n+1}}.$$

The saddlepoints are $t_1 = i$, $t_2 = -i$. The surface $\operatorname{Re} \frac{1}{2}x(t-1/t)$ has the form of ridges, separating two hollows. The saddlepoints will be found on these ridges, and therefore our contour, if it contains the point $t=0$, must pass through both saddlepoints. Using (1) we obtain:

$$\begin{aligned} J_n(x) &= \pm \frac{t^{-n-1} e^{ix}}{\sqrt{2\pi i x}} \left(1 + O\left(\frac{1}{x}\right)\right) \pm \\ &\pm \frac{(-t)^{-n-1} e^{-ix}}{\sqrt{-2\pi i x}} \left(1 + O\left(\frac{1}{x}\right)\right) = \\ &= \pm \frac{e^{i\left(x + (2n+3)\frac{\pi}{4}\right)}}{\sqrt{2\pi x}} \left(1 + O\left(\frac{1}{x}\right)\right) \pm \\ &\pm \frac{e^{-i\left(x + (2n+3)\frac{\pi}{4}\right)}}{\sqrt{2\pi x}} \left(1 + O\left(\frac{1}{x}\right)\right). \end{aligned}$$

The signs can be determined, independently of any geometric considerations, from the fact that $J_n(x)$ is real for real x , and for $x=it$, $t>0$, $J_n(x)=i_h A$, $A>0$. These considerations lead finally to:

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left[x + (2n+3)\frac{\pi}{4}\right] + O\left(\frac{e^{|Im x|}}{x^{3/2}}\right).$$

We limit ourselves here to these examples, but further on we shall repeatedly encounter applications of the saddlepoint method.

CHAPTER II

Basic Material from the Theory of Entire Functions

1. The concept of the order of growth.

The basic task of the theory of entire functions (at least from the point of view of its applications to other domains of analysis) is to establish connections between the different elements of an entire function, for example, between the coefficients, the behavior at infinity, the zeros. It would hardly be mistaken to say that the most important task of all is to establish such connections for entire functions that are in some sense regular, that is, have regularly decreasing coefficients, or regularly distributed zeros, or a simple integral representation, or a simple functional equation. However, the study of entire functions under such strong hypotheses is a very complicated task, and it is useless to expect simple and general answers. Therefore, in dealing with these and similar problems it is necessary to know those simpler and more general laws that are less exact but which hold under weaker hypotheses. This chapter is devoted to presenting just such simpler relations between the basic elements of an entire function.

Among all the elements of an entire function it is customary to single out three as the most important. These are the Taylor coefficients, the zeros of the function and, finally, its behavior at infinity.

The simplest characteristics of these elements are the rate of decrease of the coefficients, the number of zeros in the circle $|z| < r$, and the rate of growth of the logarithm of the maximum modulus of the function in the circle

$|z| \leq r$. It is customary to compare the logarithm of the maximum modulus of the function with certain very smooth functions, called *orders of growth*. More precisely:

Let $h(r)$ be a monotonic, logarithmically convex, that is

$$\frac{d \ln h(r)}{d \ln r} = \frac{r h'(r)}{h(r)} > 0$$

for $r > r_0$, twice differentiable function, satisfying the condition

$$\lim_{r \rightarrow \infty} \frac{r h'(r)}{h(r)} = \rho, \quad 0 \leq \rho \leq \infty.$$

If

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} = \sigma, \quad 0 < \sigma < \infty; \quad M(r) = \max_{|z|=r} |F(z)|,$$

then we shall say that the *entire function* $F(z)$ has the *order of growth* $h(r)$.

The number ρ is usually called the *order of the entire function* $F(z)$, and entire functions having the order of growth r^ρ are called *entire functions of order ρ and finite type σ* .

It can be shown that for every function $h(r)$ satisfying, in addition to our conditions, the condition $h(r)/\ln r \rightarrow \infty$, there exist entire functions having the order of growth $h(r)$.

The case that has been studied most is when the order of the entire function (that is, the number ρ) is different from zero and infinity. In this case it is convenient to write $h(r)$ in the form $h(r) = r^\rho l(r)$, where $r l'(r)/l(r) \rightarrow 0$ as $r \rightarrow \infty$. The function $l(r)$ is called a *slowly increasing function* (even if it is decreasing).

Let us note several properties of slowly increasing functions:

(1) For each $c > 0$

$$\lim_{r \rightarrow \infty} \frac{l(cr)}{l(r)} = 1.$$

Indeed, from the definition of a slowly increasing function we have $l'(r)/l(r) = o(1/r)$, and so

$$\ln \frac{l(cr)}{l(r)} = \int_r^{cr} \frac{l'(t)}{l(t)} dt = \int_r^{cr} o\left(\frac{1}{t}\right) dt = o(1),$$

$$\frac{l(cr)}{l(r)} = e^{o(1)} = 1 + o(1).$$

We shall say that two slowly increasing functions $l_1(r)$ and $l_2(r)$ are *equivalent*, if

$$\lim_{r \rightarrow \infty} \frac{l_1(r)}{l_2(r)} = 1.$$

(2) *The slowly increasing functions $l(r)$ and $\lambda(r)$*

$$l(r) = r \int_0^{\infty} \frac{\lambda(t) dt}{(t+r)^2} \quad (1)$$

are equivalent.

Indeed

$$\frac{l(r)}{\lambda(r)} = \frac{r}{\lambda(r)} \int_0^{\infty} \frac{\lambda(t) dt}{(t+r)^2} = \int_0^{\infty} \frac{\lambda(rt)}{\lambda(r)} \frac{dt}{(t+1)^2} \sim \int_0^{\infty} \frac{dt}{(t+1)^2} = 1.$$

In what follows we shall always assume that all slowly increasing functions have the representation (1). This means that we admit only infinitely differentiable (and, in fact, analytic) slowly increasing functions.

(3) Let $l(r)$ be slowly increasing function. Then

$$\frac{r^k l^{(k)}(r)}{l(r)} = o(1), \quad r \rightarrow \infty, \quad k \geq 1.$$

Indeed,

$$l^{(k)}(r) = (-1)^k (k+1)! r \int_0^\infty \frac{\lambda(t) dt}{(t+r)^{k+2}} + \\ + (-1)^{k-1} k! k \int_0^\infty \frac{\lambda(t) dt}{(t+r)^{k+1}}.$$

From this we find:

$$\frac{r^k l^{(k)}(r)}{l(r)} = (-1)^k (k+1)! \int_0^\infty \frac{\lambda(ru)}{l(r)} \frac{du}{(u+1)^{k+2}} + \\ + (-1)^{k-1} k! k \int_0^\infty \frac{\lambda(ru)}{l(r)} \frac{du}{(u+1)^{k+1}} = \\ = (-1)^k k! \left\{ (k+1) \int_0^\infty \frac{du}{(u+1)^{k+2}} - k \int_0^\infty \frac{du}{(u+1)^{k+1}} \right\} + o(1) = o(1).$$

(4) A slowly increasing function $l(r)$ is an analytic function, regular in the slit plane cut along the real axis from $-\infty$ to 0. Also, $\operatorname{Re} l(re^{i\theta}) \sim l(r)$, and $\operatorname{Im} l(re^{i\theta}) = o(l(r))$ for $r \rightarrow \infty$, $|\theta| \leq \pi - \eta$, $\eta > 0$.

The regularity of $l(r)$ in the plane with the slit $(-\infty, 0)$ follows at once from the integral representation. From it we obtain:

$$l(re^{i\theta}) = \int_0^\infty \lambda(t) \frac{re^{i\theta} dt}{(t+re^{i\theta})^2} = \int_0^\infty \frac{\lambda(rt) e^{i\theta}}{(t+e^{i\theta})^2} dt \sim \\ \sim \lambda(r) \int_0^\infty \frac{e^{i\theta} dt}{[(t+e^{i\theta})^2]} = \lambda(r) \sim l(r).$$

Comparing the real and imaginary parts of the equation

$$l(re^{i\theta}) = l(r) + o(l(r)),$$

we see that $\operatorname{Re} l(re^{i\theta}) \sim l(r)$, $\operatorname{Im} l(re^{i\theta}) = o(l(r))$.

Let us consider some examples of entire functions with a view to determining their order of growth.

$$1. F(z) = \int_1^{\infty} t^z e^{-t} dt.$$

We have already encountered this function in Section 5, Chapter I (example 2), and we obtained the asymptotic estimate

$$F(z) = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right) - \frac{1}{ze} \left(1 + O\left(\frac{1}{z}\right)\right).$$

by the saddlepoint method. It is easy to see that the maximum of the modulus of $F(z)$ on the circle $|z|=r$ is attained for $z=r$. Hence from our estimate we have

$$M(r) = \sqrt{2\pi r} \left(\frac{r}{e}\right)^r \left(1 + O\left(\frac{1}{r}\right)\right),$$

that is, $\ln M(r) \sim r \ln r$, which means that the order of growth of $F(z)$ is $r \ln r$.

$$2. F(z) = \int_0^{\infty} e^{zt-t} \ln t \, dt.$$

We estimated this function for $z > 0$ in Section 3, Chapter I (example 3). Also, it is easy to see that

$$M(r) = \max_{|z|=r} |F(z)| = F(r).$$

Therefore by the estimate that was obtained in Section 3, Chapter I we have $\ln M(r) \sim e^{r-1}$. Thus, the order of growth of $F(z)$ is e^r .

$$M(r) = \max_{|z|=r} |F(z)| = F(r).$$

$$3. \quad F(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\frac{n}{\rho} + 1\right)}.$$

We obtained a series of asymptotic estimates for this function in Section 4, Chapter I (example 4). Since it is clear once more that $M(r) = F(r)$, we can use the simple formula

$$F(r) = \rho e^{r^\rho} + O\left(\frac{1}{r}\right).$$

From this it is clear that

$$\ln M(r) \sim r^\rho,$$

that is, the order of growth of $F(z)$ is equal to ρ .

In conclusion we present an example of a function with the exact order of growth $h(r) = \rho l(r)$, where $l(r)$ is an arbitrary slowly increasing function. For simplicity we only consider the case $\rho > \frac{1}{2}$.

$$4. \quad F(z) = \frac{1}{2\pi i} \int_{L_{r_0}} \frac{e^{t^\rho l(t)}}{t-z} dt,$$

where L_{r_0} is the contour formed from the two rays

$$\left(r_0 e^{\frac{\pi i}{2\rho_1}}, \infty e^{\frac{\pi i}{2\rho_1}}\right), \left(\infty e^{-\frac{\pi i}{2\rho_1}}, r_0 e^{-\frac{\pi i}{2\rho_1}}\right)$$

and the circular arc: $|t| = r_0$, $|\arg t| \leq \pi/2\rho_1$, $\rho_1 < \rho$, and $r_0 > |z|$.

If the point z lies outside the angle $|\arg z| \leq \pi/2\rho_1$, then $F(z)$ is bounded. Therefore, in determining the order of growth of $F(z)$ we are only interested in the case when z lies in the angle $|\arg z| \leq \pi/2\rho$.

Since we are traversing the contour in the negative direction, the residue theorem gives, for $|\arg z| \geq \pi/2\rho$

$$F(z) = e^{z^{\rho l}(z)} + \frac{1}{2\pi l} \int_{L_0} \frac{e^{t^{\rho l}(t)}}{t-z} dt.$$

The integral is bounded. Therefore $M(r) \sim \exp(r^{\rho l}(r))$, from which we see that the order of growth of $F(z)$ is equal to $r^{\rho l}(r)$.

2. The connection between the rate of growth of an entire function and the rate of decrease of its coefficients

To solve this simplest problem of the basic problems of the theory of entire functions we need two obvious inequalities:

$$M(r) \leq \sum_{n=0}^n |a_n| r^n, \quad (1)$$

$$|a_n| \leq \frac{M(r)}{r^n}, \quad 0 < r < \infty, \quad n = 0, 1, 2, \dots \quad (2)$$

The first of these inequalities results from the fact that the modulus of a sum is less than or equal to the sum of the moduli, the second comes from Cauchy's formula for the coefficients of a power series

$$a_n = \frac{1}{2\pi l} \int_{|z|=r} \frac{F(z)}{z^{n+1}} dz.$$

Each of these inequalities is exact for the class of all entire functions. In (1) the equality is attained whenever there is a φ such that the numbers $a_n e^{in\varphi}$ all have the same argument, and in (2) it is attained if $F(z) = a_n z^n$.

It is convenient to write inequalities (1) and (2) in the form

$$\mu(r) \leq M(r) \leq \sum_{n=0}^{\infty} |a_n| r^n, \quad \mu(r) = \max_{0 \leq n < \infty} |a_n| r^n. \quad (3)$$

Let us consider the basic problem—to find estimates from above (and below) for the coefficients of the entire function $F(z)$, if we know that $M(r) \leq e^{h(r)}$ (or $M(r) \geq e^{h(r)}$). First we consider the simplest case $h(r) = r$.

THEOREM 1. *If $F(z) = \sum a_n z^n$ and $M(r) \leq Me^r$, then*

$$|a_n| \leq M \left(\frac{e}{n} \right)^n \quad (4)$$

and if $M(r_1) \geq Me^{r_1}$ for at least one value r_1 , then for at least one value n_1

$$|a_{n_1}| \geq \frac{M}{n_1!}. \quad (5)$$

Proof: To prove (4) we use inequality (2). We have:

$$|a_n| \leq \min_{0 < r < \infty} \frac{M(r)}{r^n} \leq \min_{0 < r < \infty} \frac{Me^r}{r^n} = M \left(\frac{e}{n} \right)^n.$$

To prove (5) we assume the contrary, that is, that $|a_n| < M/n!$ for all n . Then from (3) we have for all r

$$M(r) \leq \sum_{n=0}^{\infty} |a_n| r^n < \sum_{n=0}^{\infty} \frac{Mr^n}{n!} = Me^r.$$

This contradicts our hypothesis that $M(r_1) \geq Me^{r_1}$. Consequently, (5) is true.

To generalize this result to arbitrary $h(r)$, we must find two functions $A(n)$ and $a(n)$, having the following properties:

If $M(r) \leq e^{h(r)}$, then $|a_n| \leq A(n)$.

If $M(r_1) \geq \exp(h(r_1))$, then $|a_{n_1}| \geq a(n)$ for at least one n .

The function $A(n)$ can be determined precisely without too much difficulty. Indeed,

$$|a_n| \leq \min_{0 < r < \infty} \frac{M(r)}{r^n} \leq \min_{0 < r < \infty} \frac{e^{h(r)}}{r^n} = A(n).$$

For the minimum point we have the equation $(d/dr)[h(r) - n \ln r] = 0$ or $n = rh'(r)$. Let $k(n)$ denote the function inverse to $rh'(r)$; we have:

$$A(n) = \frac{e^{h(k(n))}}{[k(n)]^n}.$$

Thus we are led to the following theorem.

THEOREM 2. *If $F(z) = \sum a_n z^n$ is an entire function, and if $M(r) \leq e^{h(r)}$, where $rh'(r)$ is monotonic and if $k(n)$ is the function inverse to $rh'(r)$, then*

$$|a_n| \leq \frac{e^{h(k(n))}}{[k(n)]^n}. \quad (6)$$

We do not give the exact form of the function $a(n)$. However, we shall give an asymptotic expression for it for large n . To this end we first study the asymptotic behavior of the function

$$M_1(r) = \sum_{n=0}^{\infty} A(n) r^n$$

for large r , and then multiply $A(n)$ by a factor $T(n)$ such that

$$M_2(r) = \sum_{n=0}^{\infty} T(n) A(n) r^n \sim e^{h(r)}.$$

If we can succeed in doing this, then by repeating the reasoning of the theorem we will see that $T(n)A(n) \sim a(n)$ for large n . To find the asymptotic behaviour of $M_1(r)$ for large r we shall replace the sum

$$\sum_{n=0}^{\infty} A(n) r^n$$

by the integral

$$\int_0^{\infty} A(t) r^t dt.$$

The function $A(t)r^t$ has just a single maximum, since

$$\frac{d}{dt} \ln A(t) r^t = \frac{d}{dt} \{t \ln r + h(k(t)) - t \ln k(t)\} = \ln \frac{r}{k(t)}$$

(since $k(t)$ is the inverse function to $rh'(r)$), and $r=k(t)$ only for $t=rh'(r)$.

Hence it is clear that the difference between the sum and the integral does not exceed the magnitude of the maximal term. Thus the necessary and sufficient condition that

$$\sum_{n=0}^{\infty} A(n) r^n \sim \int_0^{\infty} A(t) r^t dt,$$

is that

$$\max_n A(n) r^n = o \left(\int_0^{\infty} A(t) r^t dt \right). \quad (7)$$

But $\max_n A(n) r^n = e^{h(r)}$, since by definition $A(n) = \min_r e^{h(r)/r^n}$. Thus, (7) can be written in the form

$$e^{h(r)} = o \left\{ \int_0^{\infty} A(t) r^t dt \right\}.$$

The Laplace method can be used to estimate this integral. We use formula (4), Section 3, Chapter I. In those notations

$$h(r, t) = \ln [A(t) r^t], \quad \varphi(t) = 1,$$

$$h_t''(r, t) = \frac{k'(t)}{k(t)}, \quad t_0(r) = rh'(r).$$

Substituting these values we obtain:

$$\begin{aligned} \int_0^\infty A(t) r^t dt &\sim A(rh'(r)) r^{rh'(r)} \sqrt{\frac{2\pi k(rh'(r))}{k'(rh'(r))}} = \\ &= e^{h(r)} \sqrt{2\pi r(rh'(r))'}. \end{aligned}$$

Condition (7) means that

$$r(rh'(r))' \rightarrow \infty, \quad (8)$$

and when it is satisfied

$$\sum_{n=0}^\infty A(n) r^n \sim e^{h(r)} \sqrt{2\pi r(rh'(r))'}.$$

(Note that from (8) it follows that $h(r)/\ln^2 r \rightarrow \infty$.) If now we put $\Upsilon(n) = [k'(n)/2\pi k(n)]^{1/2}$, then in estimating $M_2(r)$ we may repeat all the previous reasoning except that in formula (4), Section 3, Chapter I we put $\varphi(t)$, equal to $\Upsilon(t)$, instead of equal to 1, which does not affect the applicability of this formula since $\Upsilon(t) = (-h_t''(r, t)/2\pi)^{-1/2}$. We obtain

$$\sum_{n=0}^\infty \Upsilon(n) A(n) r^n \sim e^{h(r)}$$

provided that (8) is satisfied. Consequently the function $a(n)$, of which we spoke earlier, has been found asymptotically:

$$a(n) \sim \frac{e^{h(k(n))}}{[k(n)]^{1/2}} \sqrt{\frac{k'(n)}{2\pi k(n)}}, \quad (9)$$

provided that the function $h(r)$ satisfies (8).

Formula (9) enables us to formulate the following assertion:

Let $F(z) = \sum a_n z^n$ be an entire function and $M(r)$ its maximum modulus. If $M(r_k) > (1 + \epsilon) \exp(h(r_k))$ for a sequence $r_k \rightarrow \infty$, where $rh'(r)$ is monotonic and $r(rh'(r))' \rightarrow \infty$, and $k(n)$ is the function inverse to $rh'(r)$, then there exists a sequence $n_p \rightarrow \infty$ such that

$$|a_{n_p}| > \frac{e^{h(k(n_p))}}{[k(n_p)]^{n_p}} \sqrt{\frac{k'(n_p)}{2\pi k(n_p)}}. \quad (10)$$

From (6) and (10) it is easy to obtain necessary and sufficient conditions on the coefficients of a function $F(z)$ having finite order $0 < \rho < \infty$ and order of growth $h(r)$.

If

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} = \sigma, \quad \lim_{r \rightarrow \infty} \frac{rh'(r)}{h(r)} = \rho, \quad 0 < \rho < \infty,$$

then

$$\overline{\lim}_{n \rightarrow \infty} \nu(n) |a_n|^{\frac{1}{n}} = (\sigma \rho e)^{\frac{1}{\rho}}, \quad \nu(h(r)) = r.$$

Indeed, it follows from (6) that

$$|a_n| < M_* \frac{e^{(\sigma + \epsilon)h(k(n))}}{[k(n)]^n},$$

where $k(n)$ is the function inverse to $(\sigma + \epsilon)rh'(r)$. But

$$rh'(r) \sim \rho h(r), \quad (\sigma + \epsilon)rh'(r) \sim (\sigma + \epsilon)\rho h(r),$$

which means that

$$k(n) \sim \nu\left(\frac{n}{\rho(\sigma + \epsilon)}\right) \sim [\rho(\sigma + \epsilon)]^{-\frac{1}{\rho}} \nu(n)$$

$$\left(\text{since from } r \frac{h'(r)}{h(r)} \sim \rho, \text{ follows } n \frac{\nu'(n)}{\nu(n)} \sim \frac{1}{\rho}, \quad \frac{\nu(an)}{\nu(n)} \sim a^{\frac{1}{\rho}} \right).$$

Analogously

$$(\sigma + \varepsilon) h(k(n)) \sim \frac{n}{p}.$$

From which we find:

$$|a_n|^{\frac{1}{n}} < (1 + \varepsilon_1) (\rho \varepsilon e)^{\frac{1}{p}} \frac{1}{\sqrt[n]{n}}.$$

On the other hand formula (10) tells us that for an infinite sequence $n_p \rightarrow \infty$

$$|a_{n_p}| > M_1 \frac{e^{(\sigma - \varepsilon) h(k(n_p))}}{[k(n_p)]^{n_p}},$$

which gives us:

$$|a_{n_p}|^{\frac{1}{n_p}} > (1 - \varepsilon_1) (\rho \varepsilon e)^{\frac{1}{p}} \frac{1}{\sqrt[n_p]{n_p}}.$$

Comparing these two inequalities, we obtain our assertion.

One can also investigate the case when

$$\lim_{r \rightarrow \infty} r(rh'(r))' < \infty.$$

which we have put aside till now. For example, if $r(rh'(r)) \rightarrow 0$, then it is easy to see that the sum $\sum A(n)r^n$ is equivalent as $r \rightarrow \infty$ to the maximal term, that is

$$\sum_{n=0}^{\infty} A(n)r^n \sim e^{h(r)},$$

which means that $a(n) \sim A(n)$. If $\lim_{r \rightarrow \infty} r(rh'(r))' = c > 0$, then $A(n)$ and $a(n)$ differ asymptotically by a constant factor.

Our estimates for the sum of the series can be given another interesting form.

If we let $\mu(r)$ denote the maximal term of the series $\sum A(n)r^n$, and $\nu(r)$ the number of the maximal term, and

$M(r)$ the sum of the series, then writing (9) in these notations we have:

$$M(r) \sim \mu(r) \sqrt{2\pi r \nu'(r)} \quad (11)$$

(we assume that the function $\nu(r)$ has been smoothed before taking its derivative). This formula gives the connection between the growth of the maximum modulus and the decrease of the coefficients in a most compact form, for entire functions with very regular coefficients. Even though in practice it is not possible to verify these assumptions about the regularity of the coefficients, this formula is very convenient for preliminary estimates that are made without rigorous proof.

It can be shown that in the form

$$M(r_k) < (1 + \varepsilon) \mu(r_k) \sqrt{2\pi r_k \nu'(r_k)},$$

where r_k is some sequence, $r_k \rightarrow \infty$, this formula is valid for all entire functions.

Formula (11) takes an especially simple form for the case of functions of finite order ρ , $0 < \rho < \infty$. Then

$$\begin{aligned} \nu(r) &\sim r \frac{d}{dr} \ln \mu(r) = r h'(r) \sim \rho h(r) = \rho \ln \mu(r), \\ r \nu'(r) &\sim \rho \nu(r) \sim \rho^2 \ln \mu(r), \end{aligned}$$

and we have:

$$M(r) \sim \rho \mu(r) \sqrt{2\pi \ln \mu(r)}. \quad (12)$$

Similar considerations can be applied when $F(z)$ is not an entire function, but is merely regular in some finite circle.

Formulae (11) and (12) can be used in either direction, that is, to study the growth of the maximum modulus

from the coefficients, and conversely, to study the coefficients from the growth of the maximum modulus. However, the second direction can lead easily to false results. There is, however, one important case when it is possible. This is when, for all r , the maximum modulus is attained on the same ray and at no other points on the circle $|z|=r$, and when the modulus decreases in a regular manner as one moves around the circle away from this maximum point. In this case the coefficients can be estimated by the saddlepoint method, which gives the same result as (9). We now leave these general considerations and discuss two examples:

$$1. F(z) = e^{e^z}.$$

We express the coefficients of $F(z)$ by Cauchy's formula

$$a_n = \frac{1}{2\pi i} \int_C \frac{e^{e^z}}{z^{n+1}} dz.$$

The conditions for the saddlepoint method are satisfied, and it is clear that only one saddlepoint will be involved in the estimate, namely the one on the real axis. Indeed, the circle $|z|=r$ passing through the saddlepoint is already a contour on which the maximum of the modulus of the integrand is attained at the saddlepoint.

This saddlepoint is determined from the equation

$$e^z - \frac{n+1}{z} = 0.$$

Let $k(n)$ be the unique positive solution of this equation. It is not difficult to see that

$$k(n) \sim \ln n.$$

From formula (3), Section 5, Chapter I we have:

$$a_n \sim \frac{e^{e^{k(n)}}}{[k(n)]^{n+1}} \sqrt{\frac{[k(n)]^2}{2\pi(n+1)[1+k(n)]}} \sim \frac{e^{\frac{n}{\ln n}}}{[k(n)]^n \sqrt{2\pi n \ln n}}.$$

$$2. F(z) = \frac{1}{2\pi i} \int_{L_{r_0}} \frac{e^{t^\rho} dt}{t-z}, \quad \frac{1}{2} < \rho < \infty.$$

(This function is a special case of one already considered in Section 1.)

The same considerations as in example 1 show that the saddlepoint method can be applied and that in the estimate only the saddlepoint on the real axis plays a role. To find this saddlepoint recall the estimate for this function that we obtained in example 4, Section 1. We showed that for $|\arg z| \leq \pi/2\rho$

$$F(z) = e^{z^\rho} + o(1) = e^{z^\rho + o(1)} e^{-z^\rho}.$$

Therefore the saddlepoint will satisfy the equation

$$\rho z^{\rho-1} - \frac{n+1}{z} = o(z^{\rho-1} e^{-z^\rho}),$$

which gives:

$$z_0(n) = \left(\frac{n+1}{\rho}\right)^{\frac{1}{\rho}} + o(e^{-n}).$$

Using formula (2), Section 4, Chapter I again we obtain:

$$a_n \sim \frac{e^{\frac{n+1}{\rho}}}{\left(\frac{n+1}{\rho}\right)^{\frac{n+1}{\rho}}} \sqrt{\frac{[z_0(n)]^2}{2\pi[n+1+\rho(\rho-1)(z_0(n))^\rho]}} \sim \frac{e^{\frac{1}{\rho}-1}}{\sqrt{2\pi\rho n}} \left(\frac{\rho e}{n}\right)^n.$$

3. Find the asymptotic behaviour as $n \rightarrow \infty$ of the Taylor coefficients of the function

$$w = e^{z + \frac{z^2}{2} + \dots + \frac{z^m}{m}} = \sum_{n=0}^{\infty} a_n z^n.$$

By Cauchy's formula

$$a_n = \frac{1}{2\pi i} \int_C \frac{e^{z + \frac{z^2}{2} + \dots + \frac{z^m}{m}}}{z^{n+1}} dz.$$

The saddlepoint will be $z = R$, where $R > 0$ is determined by the equation

$$R + R^2 + \dots + R^m = n.$$

Applying formula (3), Section 5, Chapter I ($h(n, z) = z + \frac{1}{2}z^2 + \dots + z^m/m - n \ln z$, $\varphi(z) = 1/z$), we obtain:

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{|z|=R} e^{z + \dots + \frac{z^m}{m}} \frac{dz}{z^{n+1}} = \\ &= e^{R + \dots + \frac{R^m}{m} - n \ln R} \frac{1}{\sqrt{2\pi n m}} [1 + o(1)]. \end{aligned}$$

Let us find an explicit expression in n for the quantity in the exponent. We have:

$$R \left(\frac{1 - \frac{1}{R^m}}{1 - \frac{1}{R}} \right)^{\frac{1}{m}} = n^{\frac{1}{m}} = \zeta,$$

from which it is clear that

$$R = \zeta + b_0 + \frac{b_1}{\zeta} + \dots + \frac{b_n}{\zeta^n} + \dots$$

Therefore

$$\begin{aligned} R + \frac{R^2}{2} + \dots + \frac{R^m}{m} - n \ln R &= \\ &= R + \dots + \frac{R^m}{m} - \zeta^m \ln \frac{R}{\zeta} - \zeta^m \ln \zeta = \psi(\zeta) - \zeta^m \ln \zeta; \\ \psi(\zeta) &= c_m \zeta^m + \dots + c_1 \zeta + c_0 + \frac{c_{-1}}{\zeta} + \dots \end{aligned}$$

Let us find the coefficients of the positive powers in the series for $\Psi(\zeta)$:

$$c_k = \frac{1}{2\pi i} \oint_C \frac{\psi(\zeta)}{\zeta^{k+1}} d\zeta = \frac{1}{2\pi i} \oint_C \frac{\psi'(\zeta)}{k\zeta^k} d\zeta, \quad k > 0.$$

But $\Psi'(\zeta) = \zeta^{m-1} - m\zeta^{m-1} \ln(R/\zeta)$, from which

$$c_k = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{k\zeta^{k-m+1}} - \frac{1}{2\pi i} \oint_C \frac{m \ln \frac{R}{\zeta}}{k\zeta^{k-m+1}} d\zeta.$$

Consequently, $c_m = 1/m$, and for $k < m$

$$\begin{aligned} c_k &= -\frac{1}{2\pi i} \oint_C \frac{m \ln \frac{R}{\zeta}}{k\zeta^{k-m+1}} d\zeta = \frac{1}{2\pi i} \oint_C \frac{m}{k(m-k)} \left(\frac{R'}{R} - \frac{1}{\zeta} \right) d\zeta = \\ &= \frac{1}{2\pi i} \oint_{C_1} \frac{m}{k(m-k)} \left(\frac{1 - \frac{1}{R^m}}{1 - \frac{1}{R}} \right)^{\frac{m-k}{m}} R^{m-k-1} dR = \\ &= \frac{\left(\frac{m-k}{m} + 1 \right) \dots \left(\frac{m-k}{m} + m - 1 \right)}{k \cdot (m-k)!}. \end{aligned}$$

Similarly

$$c_0 = -\frac{1}{m} \sum_{k=2}^m \frac{1}{k}.$$

Substituting these expressions we finally obtain:

$$a_n = \frac{e^{-\frac{1}{m} \sum_{k=2}^m \frac{1}{k} + \sum_{k=1}^{m-1} \frac{\left(\frac{k}{m} + 1\right) \cdots \left(\frac{k}{m} + k - 1\right)}{k! (m-k)}} n^{1 - \frac{k}{m}}}{\sqrt{m} (2\pi n)^{\frac{m-1}{2m}} (n!)^{\frac{1}{m}}} [1 + o(1)]$$

It is not difficult to verify that in all these examples formula (9) yields the same results. This is not surprising, since a formal application of the saddlepoint method to the integral for a_n produces a formula equivalent to (9).

3. The connection between the rate of growth of an entire function and the number of zeros

The simplest characteristic of the zeros of a function $F(z)$ is the function $n(r)$, the number of zeros of $F(z)$ in the circle $|z| < r$ (each zero counted as many times as its multiplicity). Our immediate task is to find the relation between $\ln M(r)$ and $n(r)$.

To estimate $\ln M(r)$ in terms of $n(r)$ from below we use a simple formula known as *Jensen's formula*.

If $F(0) \neq 0$ and $F(z) \neq 0$ for $|z| = r$, and if $F(z)$ is regular for $|z| \leq r$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |F(re^{i\varphi})| d\varphi = \ln |F(0)| + \int_0^r \ln \frac{r}{t} dn(t). \quad (1)$$

To prove this formula we apply the mean value theorem for harmonic functions to the function $\ln |F_1(z)|$, where

$$F_1(z) = F(z) \prod_{|z_n| < r} \frac{r^2 - z\bar{z}_n}{r(z - z_n)}$$

(z_n are the zeros of $F(z)$; if z_k is a multiple zero, then we repeat it in the product as many times as its multiplicity).

The function $F_1(z)$ is regular in $|z| \leq r$ and has no zeros there. Therefore

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \ln |F_1(re^{i\varphi})| d\varphi &= \ln |F_1(0)| = \\ &= \ln |F(0)| + \sum_{|z_n| < r} \ln \frac{r}{|z_n|}. \end{aligned}$$

Therefore, writing the sum on the right side in the form of a Stieltjes integral and noting that $|F_1(z)| = |F(z)|$ for $|z| = r$, we obtain formula (1).

From (1) it is easy to estimate $\ln M(r)$ from below in terms of $n(r)$

$$\ln M(r) \geq \ln M(0) + \int_0^r \ln \frac{r}{t} dn(t). \quad (2)$$

It is in general not possible to estimate $\ln M(r)$ from above in terms of $n(r)$. Indeed, there are functions that increase arbitrarily rapidly and have no zeros whatsoever (such functions have the form $F(z) = e^{g(z)}$, where $g(z)$ is an entire function). Therefore in estimates from above we must make some auxiliary assumptions about $F(z)$ that will exclude such possibilities.

The most natural problem that can be posed in this connection is the problem of estimating the growth of the so-called *canonical products*, that is, entire functions of the form

$$P(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \frac{z^2}{2z_n^2} + \dots + \frac{z^p}{pz_n^p}}.$$

The number p is the smallest integer for which the product converges absolutely; it is called the *genus of the canonical product*. A canonical product has genus p if the series $\sum |z_n|^{-p-1}$ converges, while the series $\sum |z_n|^{-p}$ diverges.

If the zeros grow so slowly that the series $\sum |z_n|^{-p}$ diverges for all p , then for the convergence of the canonical product the number p must be replaced by an increasing function of n , and the estimation of $F(z)$ becomes much more complicated.

We shall consider in detail the estimation of canonical products of genus 0 and 1, since for them the estimates are comparatively easy and these two cases clearly illustrate the general picture.

THEOREM 1. *For a canonical product of genus 0*

$$\ln M(0) + \int_0^r \ln \frac{r}{t} dn(t) \leq \ln M(r) \leq \int_0^\infty \ln \left(1 + \frac{r}{t}\right) dn(t). \quad (3)$$

Proof: The left side of the inequality has already been established in (2). To prove the right inequality, note that the function

$$\ln |1 + te^{i\varphi}|, \quad t > 0,$$

for each fixed t takes its largest value for $\varphi = 0$. Therefore

$$\ln |F(re^{i\varphi})| \leq \sum_{n=0}^{\infty} \ln \left(1 + \frac{r}{|z_n|}\right) = \int_0^\infty \ln \left(1 + \frac{r}{t}\right) dn(t).$$

This proves our assertion.

The integral

$$\int_0^\infty \ln \left(1 + \frac{r}{t}\right) dn(t)$$

converges if $F(z)$ has genus 0, since $\ln(1+r/t) \sim r/t$ as $t \rightarrow \infty$, and

$$\int_0^{\infty} \frac{dn(t)}{t} = \sum \frac{1}{|z_n|} < \infty.$$

The situation is a little more complicated in case the canonical product $F(z)$ has genus one.

THEOREM 2. *For a canonical product of genus 1*

$$\begin{aligned} \ln M(0) + \int_0^r \ln \frac{r}{t} dn(t) &\leq \ln M(r) \leq \\ &\leq \frac{r^2}{2} \int_{\frac{r}{2}}^{\infty} \frac{dn(t)}{t^2} + \int_0^{\frac{r}{2}} \left\{ \ln \left(\frac{r}{t} - 1 \right) + \frac{r}{t} \right\} dn(t). \quad (4) \end{aligned}$$

Proof: Consider the expression

$$\ln |(1 - \alpha e^{i\varphi}) e^{\alpha e^{i\varphi}}| = \frac{1}{2} \ln (1 - 2\alpha \cos \varphi + \alpha^2) + \alpha \cos \varphi, \quad \alpha > 0.$$

It is easy to verify that for $\alpha < 2$ the maximum of this expression is attained for $\cos \varphi = \alpha/2$ and is equal to $\alpha^2/2$, and for $\alpha > 2$ the maximum is attained for $\varphi = 0$ and is equal to $\ln(\alpha - 1) + \alpha$. Therefore

$$\begin{aligned} \ln |F(re^{i\varphi})| &\leq \sum_{\frac{r}{|z_n|} > 2} \left[\ln \left(\frac{r}{|z_n|} - 1 \right) + \frac{r}{|z_n|} \right] + \sum_{\frac{r}{|z_n|} < 2} \frac{r^2}{2|z_n|^2} = \\ &= \int_0^{\frac{r}{2}} \left[\ln \left(\frac{r}{t} - 1 \right) + \frac{r}{t} \right] dn(t) + \int_{\frac{r}{2}}^{\infty} \frac{r^2}{2t^2} dn(t), \end{aligned}$$

which proves our assertion.

The integral

$$\int_0^{\infty} \frac{dn(t)}{t^2} = \sum \frac{1}{|z_n|^2}$$

converges since $F(z)$ has genus one.

Let us see what can be said about $\ln M(r)$ in case $n(r) \sim h(r) = r^\rho l(r)$ ($l(r)$ is a slowly increasing function).

First suppose that the genus of $F(z)$ is zero. As we have seen, this condition is equivalent to the convergence of the series $\sum 1/|z_n|$ or of the integrals

$$\int_0^{\infty} \frac{dn(t)}{t} = \int_0^{\infty} \frac{n(t)}{t^2} dt.$$

Consequently, either $\rho < 1$, or $\rho = 1$, and $\int_0^{\infty} l(t)/t dt < \infty$.

Integrating by parts in (3) and substituting $n(t) \sim t^\rho l(t)$ we obtain:

$$\begin{aligned} \int_0^r \ln \frac{r}{t} dn(t) &= \int_0^r \frac{n(t)}{t} dt \sim \int_0^r \frac{l(t)}{t^{1-\rho}} dt, \\ \int_0^{\infty} \ln \left(1 + \frac{r}{t}\right) dn(t) &= r \int_0^{\infty} \frac{n(t) dt}{t(t+r)} \sim r \int_0^{\infty} \frac{t^{\rho-1} l(t)}{t+r} dt. \end{aligned}$$

For $0 < \rho < 1$ this gives:

$$\left. \begin{aligned} A_1(r) &\leq \ln M(r) \leq A_2(r), \\ A_1(r) &\sim \frac{1}{\rho} r^\rho l(r), \quad A_2(r) \sim \frac{\pi}{\sin \pi \rho} r^\rho l(r), \end{aligned} \right\} \quad (5)$$

since

$$\begin{aligned} \int_0^r t^{\rho-1} l(t) dt &= r^\rho \int_0^1 u^{\rho-1} l(ur) du \sim r^\rho l(r) \int_0^1 u^{\rho-1} du = \frac{1}{\rho} r^\rho l(r), \\ r \int_0^{\infty} \frac{t^{\rho-1} l(t)}{t+r} dt &= r^\rho \int_0^{\infty} \frac{u^{\rho-1} l(ur)}{u+1} du \sim r^\rho l(r) \int_0^{\infty} \frac{u^{\rho-1} du}{u+1} = \frac{\pi}{\sin \pi \rho} r^\rho l(r). \end{aligned}$$

For $\rho = 0$ we have

$$\ln M(r) \sim l_1(r), \quad l_1(r) = \int_0^r \frac{l(t) dt}{t}, \quad (6)$$

since

$$\begin{aligned} r \int_0^\infty \frac{l(t) dt}{t(t+r)} &= \int_0^r \frac{l(t) dt}{t} - \int_0^r \frac{l(t) dt}{t+r} + r \int_r^\infty \frac{l(t) dt}{t(t+r)} \sim \\ &\sim l_1(r) - l(r) \ln 2 + l(r) \ln 2 = l_1(r). \end{aligned}$$

Finally, for $\rho = 1$ by similar reasoning we obtain:

$$\left. \begin{aligned} A_1(r) &\leq \ln M(r) \leq A_2(r), \\ A_1(r) &\sim r l(r), \quad A_2(r) \sim r \int_0^\infty \frac{l(t) dt}{t+r}. \end{aligned} \right\} \quad (7)$$

Only in one case, namely when $\rho = 0$, does $n(r) \sim h(r)$ imply $\ln M(r) \sim h_1(r)$. In all other cases $\ln M(r)$ can vary within fairly wide limits. For $0 < \rho < 1$ it was possible to assert at least that

$$\frac{1}{\rho} \leq \lim_{r \rightarrow \infty} \frac{\ln M(r)}{n(r)} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{n(r)} \leq \frac{\pi}{\sin \pi \rho}, \quad n(r) \sim r^\rho l(r),$$

that is to find at least the order of $M(r)$; in case $\rho = 1$ even this is impossible, since

$$\frac{A_2(r)}{r l(r)} \sim \int_0^\infty \frac{l(ur)}{l(r)} \frac{du}{u+1} \rightarrow \int_0^\infty \frac{du}{u+1} = \infty.$$

The asymptotic behaviour of the functions

$$l_2(r) = \int_0^\infty \frac{l(t) dt}{t+r} \quad \text{and} \quad l_1(r) = \int_0^r \frac{l(t) dt}{t}$$

can not be given by a single formula valid for all slowly

increasing functions $l(t)$. In general one can merely assert that

$$l(r) = o(l_i(r)), \quad l_i(r) \leq O(l(r) \ln r), \quad i = 1, 2.$$

If one makes some more restrictive assumption about $l(r)$, for example, $l(r) = (\ln r)^\alpha l_0(\ln r)$ (where $l_0(r)$ is slowly increasing), then asymptotic expressions can be obtained for $l_1(r)$ and $l_2(r)$ (the first for $\alpha > 0$, the second for $\alpha < -1$)

$$l_1(r) \sim \frac{l(r) \ln r}{\alpha + 1}, \quad l_2(r) \sim -\frac{l(r) \ln r}{\alpha + 1}.$$

Note also that if we are merely interested in

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h_1(r)},$$

then in place of (5), (6), (7) we have the following result:

THEOREM 3. *Let the canonical product $F(z)$ have genus zero, and let $\limsup n(r)/h(r) = \alpha$. If $\lim rh'(r)/h(r) = 0$, then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h_1(r)} = \alpha, \quad h_1(r) = \int_1^r \frac{h(t)}{t} dt; \quad (8)$$

if $\lim rh'(r)/h(r) = \rho$, $0 < \rho < 1$, then

$$\frac{\alpha}{\rho} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} \leq \frac{\pi \alpha}{\sin \pi \rho}; \quad (9)$$

if $\lim rh'(r)/h(r) = 1$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} \geq \alpha, \quad \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h_2(r)} \leq \alpha, \quad h_2(r) = r \int_1^\infty \frac{h(t) dt}{t(t+r)}. \quad (10)$$

Now let us consider the case when the genus of the canonical product is one. If we again put $n(r) \sim r^\rho l(r)$, then there are three possibilities. The first, $1 < \rho < 2$, the second, $\rho = 1$ and the integral $\int_1^\infty (l(t)/t) dt = \infty$, the third, $\rho = 2$ and the integral $\int_1^\infty (l(t)/t) dt < \infty$.

For $1 < \rho < 2$ we have:

$$\left. \begin{aligned} A_1(r) &\leq \ln M(r) \leq A_2(r), \\ A_1(r) &\sim \frac{r^\rho l(r)}{\rho}, \quad A_2(r) \sim c_1(\rho) r^\rho l(r); \\ c_1(\rho) &= \frac{2^{2-\rho}}{2-\rho} + \int_0^{\frac{1}{2}} \frac{u^{\rho-2} du}{1-u}, \end{aligned} \right\} \quad (11)$$

since

$$\begin{aligned} &\int_0^{\frac{r}{2}} \left\{ \ln \left(\frac{r}{t} - 1 \right) + \frac{r}{t} \right\} dn(t) + \int_{\frac{r}{2}}^{\infty} \frac{r^2}{2t^2} dn(t) = \\ &= r^2 \left\{ \int_0^{\frac{r}{2}} \frac{n(t) dt}{t^2(r-t)} + \int_{\frac{r}{2}}^{\infty} \frac{n(t)}{t^2} dt \right\} \sim \\ &\sim r^\rho l(r) \left\{ \int_{\frac{1}{2}}^{\infty} \frac{dt}{u^{2-\rho}} + \int_0^{\frac{1}{2}} \frac{u^{\rho-2} du}{1-u} \right\} = c_1(\rho) r^\rho l(r). \end{aligned}$$

For $\rho = 1$ we have:

$$\left. \begin{aligned} A_1(r) &\leq \ln M(r) \leq A_2(r), \\ A_1(r) &\sim rl(r), \quad A_2(r) \sim rl_1(r); \\ l_1(r) &= \int_0^r \frac{l(t)}{t} dt. \end{aligned} \right\} \quad (12)$$

Indeed,

$$A_2(r) = r^2 \left\{ \int_0^{\frac{r}{2}} \frac{n(t) dt}{t^2(r-t)} + \int_{\frac{r}{2}}^{\infty} \frac{n(t) dt}{t^3} \right\} \sim$$

$$\sim r^2 \int_0^{\frac{r}{2}} \frac{l(t) dt}{t(r-t)} + r^2 \int_{\frac{r}{2}}^{\infty} \frac{l(t) dt}{t^3}.$$

But

$$r^2 \int_{\frac{r}{2}}^{\infty} \frac{l(t) dt}{t^3} \sim 2rl(r),$$

$$r^2 \int_0^{\frac{r}{2}} \frac{l(t) dt}{t(r-t)} = r \int_0^{\frac{r}{2}} \frac{l(t) dt}{t} + r \int_0^{\frac{r}{2}} \frac{l(t) dt}{r-t} =$$

$$= r \int_0^{\frac{r}{2}} \frac{l(t) dt}{t} - r \int_{\frac{r}{2}}^r \frac{l(t) dt}{t} + r \int_0^{\frac{r}{2}} \frac{l(t) dt}{r-t} \sim r \int_0^{\frac{r}{2}} \frac{l(t) dt}{t} = rl_1(r),$$

and $l(r) = o(l_1(r))$. Therefore $A_2(r) \sim rl_1(r)$.

In exactly the same way for $\rho = 2$

$$A_1(r) \sim \frac{r^2}{2} l(r), \quad A_2(r) \sim r^2 l_2(r), \quad l_2(r) = \int_0^{\infty} \frac{l(t) dt}{t+r}. \quad (13)$$

The proof of these inequalities is entirely similar.

In exactly the same way one can prove a theorem analogous to Theorem 3.

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After this discussion of canonical products of genus 0 and 1 it is completely clear what sort of results can be obtained for arbitrary genus.

Indeed, the fundamental thing is to obtain inequalities of the type of (3) and (4). When the genus was 0 or 1 we obtained sharp inequalities. However, in the discussion that followed this sharpness was not needed. It is not too difficult to obtain crude inequalities analogous to (3) and (4) in the general case. Indeed, let us estimate the maximum

$$\begin{aligned} \ln \left| (1 - a e^{i\varphi}) e^{a^2 e^{i\varphi} + \frac{a^3}{2} e^{2i\varphi} + \dots + \frac{a^p}{p} e^{p i\varphi}} \right| = \\ = \frac{1}{2} \ln (1 - 2a \cos \varphi + a^2) + a \cos \varphi + \\ + \frac{a^3}{2} \cos 2\varphi + \dots + \frac{a^p}{p} \cos p\varphi. \end{aligned}$$

For $0 < a \leq \alpha \leq b < \infty$ this maximum is a bounded function of α . For $\alpha \rightarrow 0$ this maximum behaves like $\alpha^{p+1}/(p+1)$, since that is the first term in the expansion of the function

$$\ln(1 - \alpha) + \alpha + \frac{\alpha^2}{2} + \dots + \frac{\alpha^p}{p}$$

in powers of α . For $\alpha \rightarrow \infty$ this maximum behaves like α^p/p . Therefore

$$\begin{aligned} \ln M(r) \leq (1 + \epsilon) \int_0^{k_1 r} \frac{r^p}{\rho t^p} dn(t) + \\ + (1 + \epsilon) \int_{k_2 r}^{\infty} \frac{r^{p+1}}{(p+1) t^{p+1}} dn(t) + C_n(k_2 r), \quad (14) \end{aligned}$$

where the constants k_1 , k_2 and C depend only on ϵ .

Using (14) and arguments similar to those used before we obtain the following theorem.

THEOREM 4. *Let $F(z)$ be a canonical product of genus p . If $\limsup n(r)/h(r) = \alpha$, $rh'(r)/h(r) \rightarrow \rho$ and if ρ is not an integer, then*

$$\frac{\alpha}{\rho} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} \leq \alpha C(\rho). \quad (15)$$

($C(\rho)$ is a constant, depending only on ρ .) If $\rho = p+1$, then

$$\left. \begin{aligned} \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} &\geq \frac{\alpha}{\rho}, & \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h_1(r)} &\leq \alpha, \\ h_1(r) &= \int_0^r \frac{h(t)}{t^{\rho+1}} dt. \end{aligned} \right\} \quad (16)$$

If $\rho = p$, then

$$\left. \begin{aligned} \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} &\geq \frac{\alpha}{\rho}, & \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h_2(r)} &\leq \alpha, \\ h_2(r) &= r^\rho \int_0^\infty \frac{h(t) dt}{t^\rho (t+r)}. \end{aligned} \right\} \quad (17)$$

The exact value of the constant $C(\rho)$ we calculated earlier for $0 < \rho < 2$: $C(\rho) = \pi / \sin \pi \rho$ for $0 < \rho < 1$, while for $1 < \rho < 2$

$$C(\rho) = \frac{2^{2-\rho}}{2-\rho} + \int_0^{\frac{1}{2}} \frac{u^{\rho-2}}{1-u} du.$$

To conclude this section we shall solve a problem that is naturally suggested by the results we have obtained, namely the question of determining the order of an entire function of integer order by its zeros.

THEOREM 5. Let $F(z)$ be a canonical product of genus p with $\lim n(r)/h(r) = \alpha > 0$, $\lim rh'(r)/h(r) = \rho > 0$, where ρ is an integer. Let

$$\begin{aligned} \nu(r) &= \left| \sum_{|z_n| < r} z_n^{-\rho} \right| & \text{npu } \rho = p, \\ \nu(r) &= \left| \sum_{|z_n| > r} z_n^{-\rho} \right| & \text{npu } \rho = p + 1. \end{aligned}$$

Then

$$\ln M(r) = \frac{r^\rho \nu(r)}{\rho} + O(h(r)).$$

In other words, the order of $\ln M(r)$ is equal to the larger of the two quantities $r^\rho \nu(r)$ and $h(r)$.

Proof: We have

$$\ln F(z) = \sum_{n=0}^{\infty} \left\{ \ln \left(1 - \frac{z}{z_n} \right) + \frac{z}{z_n} + \frac{z^2}{2z_n^2} + \dots + \frac{z^p}{pz_n^p} \right\}.$$

Let us assume now, for definiteness, that $\rho = p$. Then the series $\sum |z_n|^{-\rho}$ diverges. We have:

$$\begin{aligned} & \left| \ln F(z) - \frac{z^p}{p} \sum_{|z_n| < r} z_n^{-p} \right| \leq \\ & \leq \left| \sum_{|z_n| < r} \ln \left| \left(1 - \frac{z}{z_n} \right) \exp \left(\frac{z}{z_n} + \dots + \frac{z^p}{pz_n^p} \right) \right| \right| + \\ & + \left| \sum_{|z_n| < r} \ln \left| \left(1 - \frac{z}{z_n} \right) \exp \left(\frac{z}{z_n} + \dots + \frac{z^{p-1}}{(p-1)z_n^{p-1}} \right) \right| \right|. \quad (18) \end{aligned}$$

But by the same considerations as in the proof of (14),

$$\left| \sum_{|z_n| > r} \ln \left| \left(1 - \frac{z}{z_n} \right) \exp \left(\frac{z}{z_n} + \dots + \frac{z^p}{p z_n^p} \right) \right| \right| \leqslant$$

$$\leqslant C_1 n(k_1 r) + (1 + \varepsilon) r^{p+1} \int_r^\infty \frac{dn(t)}{(p+1)t^{p+1}}. \quad (19)$$

$$\left| \sum_{|z_n| < r} \ln \left| \left(1 - \frac{z}{z_n} \right) \exp \left(\frac{z}{z_n} + \dots + \frac{z^{p-1}}{(p-1)z_n^{p-1}} \right) \right| \right| \leqslant$$

$$\leqslant C_2 n(k_2 r) + (1 + \varepsilon) r^{p-1} \int_0^r \frac{dn(t)}{(p-1)t^{p-1}}. \quad (20)$$

Substituting the estimates (19) and (20) into (18) we obtain:

$$\left| \ln F(z) - \sum_{|z_n| < r} \frac{z^p}{p z_n^p} \right| \leqslant$$

$$\leqslant Cn(kr) + 2r^{p+1} \int_r^\infty \frac{dn(t)}{(p+1)t^{p+1}} + 2r^{p-1} \int_0^r \frac{dn(t)}{(p-1)t^{p-1}}$$

Using the fact that $n(r) < (\alpha + \varepsilon)h(r)$ and $h(kr) < (k^p + \varepsilon)h(r)$ for $r > r_0(\varepsilon)$, we have:

$$\left| \ln F(z) - \frac{z^p}{p} \sum_{|z_n| < r} \frac{1}{z_n^p} \right| \leqslant$$

$$\leqslant C'h(r) + 2r^{p+1} \int_r^\infty \frac{h(t)}{t^{p+2}} dt + 2r^{p-1} \int_0^r \frac{h(t)}{t^p} dt.$$

But $p = \bar{p}$, and so $h(t) = t^p l(t)$,

$$r^{p-1} \int_0^r \frac{h(t)}{t^p} dt \sim h(r), \quad r^{p+1} \int_r^\infty \frac{h(t)}{t^{p+2}} dt \sim h(r),$$

so that

$$\left| \ln F(z) - \frac{z^p}{p} \sum_{|z_n| < r} z_n^{-p} \right| = O(h(r))$$

or

$$\ln |F(re^{i\varphi})| = \operatorname{Re} \frac{z^p}{p} \sum_{|z_n| < r} z_n^{-p} + O(h(r)), \quad (21)$$

from which $\ln M(r) = (r^\rho/\rho)\nu(r) + O(h(r))$. This gives us the assertion of the theorem for the case $\rho = p$.

The proof for the case $\rho = p+1$ is almost exactly the same, except that we use

$$\frac{z^p}{p} \sum_{|z_n| > r} z_n^{-p},$$

instead of

$$\frac{z^p}{p} \sum_{|z_n| < r} z_n^{-p},$$

and in place of (21) we obtain the formula

$$\ln |F(re^{i\varphi})| = \operatorname{Re} \frac{z^p}{p} \sum_{|z_n| > r} z_n^{-p} + O(h(r)). \quad (22)$$

4. Entire functions with regularly distributed zeros

The results obtained in the preceding section may give the false impression that a precise knowledge of the zeros does not enable one to make very precise statements about the modulus of the function. The reason that our

results were not very precise is that the function $n(r)$ does not sufficiently characterize the distribution of the zeros.

Here we wish to consider how accurately we can describe the growth of the function and of its coefficients if we know the zeros and if these zeros are very regularly distributed. First we obtain an estimate for the modulus of a canonical product of genus zero.

THEOREM 1. *Let*

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\lambda(n)}\right),$$

where $\lambda(n) = n^{1/\rho} l(n)$, $0 \leq \rho \leq 1$, and $l(n)$ is slowly increasing. Let $v(t)$ be the function inverse to $\lambda(n)$. For $r = |z| \rightarrow \infty$ we have the estimate

$$\ln F(z) = z \int_{\lambda(1)}^{\infty} \frac{v(t) dt}{t(t+z)} + \frac{1}{2} \ln \left(1 + \frac{z}{\lambda(1)}\right) + C + O\left(\frac{1}{v(r)}\right), \quad (1)$$

uniformly in z in the angle $|\arg z| \leq \pi - \eta$, $\eta > 0$. Here C is the constant

$$C = \lim_{n \rightarrow \infty} \left\{ \int_1^n \ln \lambda(x) dx - \sum_{k=1}^n \ln \lambda(k) + \frac{1}{2} \ln \lambda(n) \right\}.$$

Proof. Since

$$\ln F(z) = \sum_{n=1}^{\infty} \ln \left(1 + \frac{z}{\lambda(n)}\right),$$

resembles the approximating sums for an integral, we can attempt to use the method demonstrated in example 4, Section 1, Chapter I to estimate $\ln F(z)$. We have:

$$\begin{aligned}\ln F(z) &= \int_1^{\infty} \ln \left(1 + \frac{z}{\lambda(x)} \right) d[x] = \int_1^{\infty} \ln \left(1 + \frac{z}{\lambda(x)} \right) dx + \\ &+ \int_1^{\infty} \ln \left(1 + \frac{z}{\lambda(x)} \right) d \left([x] - x + \frac{1}{2} \right) = I_1 + I_2.\end{aligned}$$

First we transform the principal term I_1 . We have:

$$\begin{aligned}I_1 &= \ln \left(1 + \frac{z}{\lambda(1)} \right) + z \int_1^{\infty} \frac{x \lambda'(x) dx}{\lambda(x)(z + \lambda(x))} = \\ &= \ln \left(1 + \frac{z}{\lambda(1)} \right) + z \int_{\lambda(1)}^{\infty} \frac{\nu(t) dt}{t(z + t)}. \quad (2)\end{aligned}$$

We break the integral I_2 into three parts:

$$\begin{aligned}I_2 &= I'_2 + I''_2 + I'''_2 = \int_1^{[\nu(r)]} \ln(\lambda(x) + z) d \left([x] - x + \frac{1}{2} \right) - \\ &- \int_1^{[\nu(r)]} \ln \lambda(x) d \left([x] - x + \frac{1}{2} \right) + \\ &+ \int_{[\nu(r)]}^{\infty} \ln \left(1 + \frac{z}{\lambda(x)} \right) d \left([x] - x + \frac{1}{2} \right).\end{aligned}$$

Consider I'_2 . We have:

Since

$$\int_a^b \left([x] - x + \frac{1}{2} \right)$$

is bounded for arbitrary a and b , we have

$$\begin{aligned} \int_1^{[\nu(r)]} \frac{\lambda'(x)}{\lambda(x)+z} \left([x] - x + \frac{1}{2} \right) dx &= O \left\{ \int_1^{[\nu(r)]} \left| \frac{d}{dx} \frac{\lambda'(x)}{\lambda(x)+z} \right| dx \right\} = \\ &= O \left\{ \frac{\lambda'(1)}{\lambda(1)+r} + \frac{\lambda'(\nu(r))}{\lambda(\nu(r))+r} \right\} = O \left\{ \frac{1}{r} + \frac{1}{r\nu(r)} \right\} = \\ &= O \left(\frac{1}{r} + \frac{1}{\nu(r)} \right) = O \left(\frac{1}{\nu(r)} \right), \end{aligned}$$

since $r\nu'(r) \sim \rho\nu(r)$. This means that

$$I'_2 = \frac{1}{2} \ln \frac{z + \lambda([\nu(r)])}{z + \lambda(1)} + O \left(\frac{1}{\nu(r)} \right). \quad (3)$$

In exactly the same way

$$I''_2 = -\frac{1}{2} \ln \frac{z + \lambda([\nu(r)])}{\lambda([\nu(r)])} + O \left(\frac{1}{\nu(r)} \right). \quad (4)$$

Finally,

$$I''_2 = \int_1^{[\nu(r)]} \ln \lambda(x) dx - \sum_{n=1}^{[\nu(r)]} \ln \lambda(n),$$

and by the Euler-Maclaurin formula (formula (12) Section 1, Chapter I) we obtain:

$$\begin{aligned} I''_2 &= C - \frac{1}{2} \ln \lambda([\nu(r)]) + O \left(\frac{\lambda'(\nu(r))}{\lambda(\nu(r))} \right) = \\ &= C - \frac{1}{2} \ln \lambda([\nu(r)]) + O \left(\frac{1}{\nu(r)} \right). \quad (5) \end{aligned}$$

Comparing (2), (3), (4) and (5) we obtain the theorem.

One could obtain all the terms in the expansion by proceeding in this manner, but we prefer to obtain them by another method in Chapter III.

The case of higher genus is somewhat more complicated. Since there is nothing essentially new, and the calculations become more involved, we shall consider in detail only the case of genus one.

THEOREM 2. *Let*

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\lambda(n)}\right) e^{-\frac{z}{\lambda(n)}},$$

where $\lambda(n) = n^{1/\rho} l(n)$, $1 \leq \rho \leq 2$, and $l(n)$ is slowly increasing. Let $\nu(t)$ be the inverse function to $\lambda(n)$. Then for $r \rightarrow \infty$ we have the estimate

$$\begin{aligned} \ln F(z) = & -z^2 \int_{\lambda(1)}^{\infty} \frac{\nu(t) dt}{t^2(t+z)} + \left(C_1 - \frac{1}{\lambda(1)}\right)z + \\ & + \frac{1}{2} \ln(z + \lambda(1)) + C + O\left(\frac{1}{r}\right), \quad (6) \end{aligned}$$

which holds uniformly for z in the angle $|\arg z| \leq \pi - \eta$, $\eta > 0$.

Here

$$\begin{aligned} C = \lim_{n \rightarrow \infty} \left\{ \int_1^n \ln \lambda(x) dx - \sum_{k=1}^n \ln \lambda(k) + \frac{1}{2} \ln \lambda(n) \right\}, \\ C_1 = \lim_{n \rightarrow \infty} \left\{ \int_1^n \frac{dx}{\lambda(x)} - \sum_{k=1}^n \frac{1}{\lambda(k)} \right\}. \end{aligned}$$

Proof: First,

$$\begin{aligned}\ln F(z) &= \int_1^{\infty} \left\{ \ln \left(1 + \frac{z}{\lambda(x)} \right) - \frac{z}{\lambda(x)} \right\} d[x] = \\ &= \int_1^{\infty} \left\{ \ln \left(1 + \frac{z}{\lambda(x)} \right) - \frac{z}{\lambda(x)} \right\} dx + \\ &+ \int_1^{\infty} \left\{ \ln \left(1 + \frac{z}{\lambda(x)} \right) - \frac{z}{\lambda(x)} \right\} d \left([x] - x + \frac{1}{2} \right) = I_1 + I_2.\end{aligned}$$

Let us transform I_1 . We have:

$$\begin{aligned}I_1 &= \ln \left(1 + \frac{z}{\lambda(1)} \right) - \frac{z}{\lambda(1)} + \\ &+ z \int_1^{\infty} \left\{ \frac{1}{\lambda(x)(z + \lambda(x))} - \frac{1}{(\lambda(x))^2} \right\} x \lambda'(x) dx = \\ &= -z^2 \int_{\lambda(1)}^{\infty} \frac{v(t) dt}{t^2(t+z)} - \frac{z}{\lambda(1)} + \ln \left(1 + \frac{z}{\lambda(1)} \right). \quad (7)\end{aligned}$$

We break the integral I_2 into three parts:

$$\begin{aligned}I_2 &= I_2' + I_2'' + I_2''' = \int_1^{[v(r)]} \ln(z + \lambda(x)) d \left([x] - x + \frac{1}{2} \right) + \\ &+ \int_{[v(r)]}^{\infty} \left\{ \ln \left(1 + \frac{z}{\lambda(x)} \right) - \frac{z}{\lambda(x)} \right\} d \left([x] - x + \frac{1}{2} \right) - \\ &- \int_1^{[v(r)]} \left\{ \ln \lambda(x) + \frac{z}{\lambda(x)} \right\} d \left([x] - x + \frac{1}{2} \right).\end{aligned}$$

Just as in the preceding theorem we obtain for I_2' and I_2'' , the estimates

$$I_2' = \frac{1}{2} \ln \frac{z + \lambda([\nu(r)])}{z + \lambda(1)} + O\left(\frac{1}{r}\right), \quad (8)$$

$$I_2'' = -\frac{1}{2} \left\{ \ln \frac{z + \lambda([\nu(r)])}{\lambda([\nu(r)])} - \frac{z}{\lambda([\nu(r)])} \right\} + O\left(\frac{1}{\nu(r)}\right). \quad (9)$$

For I_2''' , we obtain from the Euler-Maclaurin formula:

$$\begin{aligned} I_2''' &= \int_1^{[\nu(r)]} \ln \lambda(x) dx - \sum_{n=1}^{[\nu(r)]} \ln \lambda(n) + z \left\{ \int_1^{[\nu(r)]} \frac{dx}{\lambda(x)} - \sum_{n=1}^{[\nu(r)]} \frac{1}{\lambda(n)} \right\} = \\ &= C - \frac{1}{2} \ln \lambda([\nu(r)]) + zC_1 - \frac{z}{2\lambda([\nu(r)])} + O\left(\frac{1}{\nu(r)}\right) + O\left(\frac{r\lambda'(\nu(r))}{(\lambda(\nu(r)))^2}\right) = \\ &= C + zC_1 - \frac{1}{2} \ln \lambda([\nu(r)]) - \frac{z}{2\lambda([\nu(r)])} + O\left(\frac{1}{\nu(r)}\right). \quad (10) \end{aligned}$$

Combining (7), (8), (9) and (10) and noting that $r=0(\nu(r))$ we obtain the assertion of the theorem.

After these two theorems it is easy to see the form of the formulae for functions of higher genus. Let us consider the order of the remainder term in this case. The remainder term arises either from estimating the integrals of the second derivative, or from applying the Euler-Maclaurin formula. In the first case it is $O(1/r)$, in the second it is

$$O\left(\frac{r^m \lambda'(\nu(r))}{(\lambda(\nu(r)))^{m+1}}\right).$$

But $\nu(r) = r^\rho l(r)$, and so $r^m \lambda'[\nu(r)] / \{\lambda[\nu(r)]\}^m = O(1/r)$ for $\rho > 1$, and thus the remainder term is of the order $O(1/r)$. Thus, if

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\lambda(n)}\right) \exp \sum_{m=1}^p \frac{(-1)^m z^m}{m [\lambda(n)]^m},$$

then for $r \rightarrow \infty$, $|\arg z| \leq \pi - \eta$, $\eta > 0$ (where $\nu(t)$ is the inverse function to $\lambda(n)$) we have

$$\ln F(z) = (-1)^p z^{p+1} \int_{\lambda(1)}^{\infty} \frac{\nu(t) dt}{t^{p+1}(t+z)} + \\ + \sum_{m=1}^p \left(C_m - \frac{1}{(\lambda(1))^m} \right) z^m + C + \frac{1}{2} \ln \left(1 + \frac{z}{\lambda(1)} \right) + O\left(\frac{1}{r}\right) \quad (11)$$

Here

$$C_m = \lim_{n \rightarrow \infty} \left\{ \int_1^n \frac{dx}{(\lambda(x))^m} - \sum_{k=1}^n \frac{1}{(\lambda(k))^m} \right\}, \\ C = \lim_{n \rightarrow \infty} \left\{ \int_1^n \ln \lambda(x) dx - \sum_{k=1}^n \ln \lambda(k) + \frac{1}{2} \ln \lambda(n) \right\}.$$

Let us say a few words about the leading term in (1), (6) and (11). It is somewhat strange that in the asymptotic formula there is a term expressed in terms of the original function in a rather complicated manner. The fact is that the function

$$\nu_1(z) = (-1)^p z^{p+1} \int_{\lambda(1)}^{\infty} \frac{\nu(t) dt}{t^{p+1}(t+z)}$$

does not have a good asymptotic expression in terms of $\nu(t)$ for arbitrary $\nu(t)$. This is reminiscent of the fact that $\ln z$ does not have an asymptotic expression, for large z , in terms of powers of z . However, if $\nu(t) = t^\rho$ (ρ is not an integer), then $\nu_1(z)$ can be expressed quite precisely, in fact the following asymptotic formula is valid

$$\nu_1(z) \sim \frac{\pi z^\rho}{\sin \pi \rho} \left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \right) (|\arg z| \leq \pi - \eta, \eta > 0) \quad (12)$$

If $\lim r\nu'(r)/\nu(r) = \rho$ and ρ is not an integer then the asymptotic formula

$$v_1(re^{i\varphi}) \sim \frac{\pi\nu(r)e^{i\rho\varphi}}{\sin \pi\rho} \quad (13)$$

is valid for $|\arg z| \leq \pi - \eta$, $\eta > 0$. However, it is in general impossible to give the remainder term.

From (11) and (13) the following qualitative conclusion can be reached—for even p the direction of greatest growth of the canonical product is the direction opposite to the direction of the zeros, while for odd p it is the same direction as the zeros. The determination of the constants C and C_m in formula (11) for concrete functions $\lambda(n)$ is very complicated as a rule. With the aid of (11) we can easily obtain asymptotic estimates for canonical products having zeros in k different directions. For example, let

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{ze^{\frac{\pi i}{2p}}}{\lambda(n)}\right) \left(1 - \frac{ze^{-\frac{\pi i}{2p}}}{\lambda(n)}\right) \times \\ \times \exp \left\{ \sum_{m=1}^p \frac{z^m}{m(\lambda(n))^m} \left(e^{\frac{\pi i m}{2p}} + e^{-\frac{\pi i m}{2p}} \right) \right\}.$$

If we introduce the function $F_0(z)$

$$F_0(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\lambda(n)}\right) \exp \sum_{m=1}^p (-1)^m \frac{z^m}{m(\lambda(n))^m}, \quad (14)$$

then we can represent $F(z)$ in the form

$$F(z) = F_0\left(-ze^{\frac{\pi i}{2p}}\right) F_0\left(-ze^{-\frac{\pi i}{2p}}\right).$$

Applying (11) to $\ln F_0(z)$ we obtain:

$$\begin{aligned} \ln F(z) = & -2z^{p+1} \int_{\lambda(1)}^{\infty} \frac{t \cos \frac{(p+1)\pi}{2\rho} - z \cos \frac{\pi p}{2\rho}}{t^{p+1} \left(t^2 - 2tz \cos \frac{\pi}{2\rho} + z^2 \right)} \nu(t) dt + \\ & + 2 \sum_{m=1}^p z^m \left(C_m - \frac{1}{m} (\lambda(1))^{-m} \right) \cos \frac{m\pi}{2\rho} + \\ & + 2C + \ln \frac{z}{\lambda(1)} + O\left(\frac{1}{r}\right) \quad (15) \end{aligned}$$

for z lying outside the angles $|\arg z + \pi/2\rho| < \eta$.

In particular, if $\rho = p$, $\nu(t) = t^p$

$$\begin{aligned} \ln F(z) = & 2z^{p+1} \sin \frac{\pi}{2\rho} \int_1^{\infty} \frac{dt}{t^2 - 2zt \cos \frac{\pi}{2\rho} + z^2} + \\ & + 2 \sum_{m=1}^{p-1} \left(C_m - \frac{1}{m} \right) z^m \cos \frac{m\pi}{2\rho} + 2C + \ln z + O\left(\frac{1}{r}\right). \end{aligned}$$

Transforming this expression we obtain finally for $|\arg z| \leq \pi/2\rho - \eta$.

$$\begin{aligned} F(z) = & \frac{\pi z^p}{\rho} + 2 \sum_{m=1}^{p-1} \left(C_m - \frac{1}{m} + \frac{1}{\rho - m} \right) z^m \cos \frac{m\pi}{2\rho} + \ln z + \\ & + 2 \left(C + \frac{1}{\rho} \right) + O\left(\frac{1}{z}\right) \quad (16) \end{aligned}$$

$$\begin{aligned} F(z) = & -\pi \left(2 - \frac{1}{\rho} \right) z^p + \\ & + 2 \sum_{m=1}^{p-1} \left(C_m - \frac{1}{m} + \frac{1}{\rho - m} \right) z^m \cos \frac{m\pi}{2\rho} + \\ & + \ln z + 2 \left(C + \frac{1}{\rho} \right) + O\left(\frac{1}{z}\right) \quad (17) \end{aligned}$$

From these estimates it is clear that the more regularly the zeros of the entire function are distributed, the more precisely we can determine its asymptotic behavior.

Hence it is natural to expect that in such cases it will be possible to find asymptotic estimates for the coefficients also. And this is indeed the case. In case the direction of greatest growth of the function is different from the direction in which the zeros are distributed, there is no difficulty in applying the saddlepoint method. In this case the estimate for the coefficients is almost exactly the same as the estimates obtained for the coefficients of the functions considered in the examples of Section 1, where the functions were given explicitly. It is somewhat more complicated when the direction of greatest growth coincides with the direction in which the zeros are distributed. Without entering into a detailed discussion of this case, we consider one example of this sort.

$$1. \quad \frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

Let us note first of all that

$$\begin{aligned} \frac{1}{\Gamma(z)\Gamma(1-z)} &= \frac{1}{z\Gamma(z)\Gamma(1-z)} \\ &= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{1}{\pi} \sin \pi z. \end{aligned}$$

Therefore $1/\Gamma(z)$ is given by

$$\frac{1}{\Gamma(z)} = \frac{1}{\pi} \sin \pi z \Gamma(1-z).$$

To determine the coefficients of $1/\Gamma(z)$ we apply the saddlepoint method to the integral

$$a_n = \frac{1}{2\pi i} \int_C \frac{dz}{z^{n+1} \Gamma(z)} = \frac{1}{2\pi i} \int_C \sin \pi z \Gamma(1-z) \frac{dz}{z^{n+1}},$$

where we substitute into formula (3), Section 4, Chapter I:

$$\varphi(z) = \frac{i \sin \pi z}{\pi}, \quad h(z, n) = \ln \Gamma(1-z) - (n+1) \ln z.$$

It is clear that the saddlepoint method can be applied, and that the only saddlepoint that plays a role in the estimate is the one lying on the negative real axis. Let us find this point. From Stirling's formula

$$\ln \Gamma(z+1) = z \ln z - z + \frac{1}{2} \ln z + \frac{1}{2} \ln 2\pi + O\left(\frac{1}{r}\right).$$

Let us write the equation for the saddlepoint, putting $z = -x$,

$$\ln x - \frac{1}{2x} + \frac{n+1}{x} = O\left(\frac{1}{x^2}\right), \quad x \ln x = n + \frac{1}{2} + O\left(\frac{1}{x}\right).$$

If $\mu(n)$ denotes the inverse function to $x \ln x$, then we obtain:

$$x_0(n) = \mu\left(n + \frac{1}{2}\right) + O\left(\frac{\ln n}{n}\right), \quad x_0(n) \sim \frac{n}{\ln n}.$$

In addition

$$h''_s(n, -x_0(n)) = -\frac{1}{x_0(n)} - \frac{n + \frac{1}{2}}{(x_0(n))^2} +$$

$$O\left(\frac{1}{(x_0(n))^3}\right) \sim -\frac{(\ln n)^2}{n}.$$

Thus

$$a_n = \sqrt{\frac{n}{2\pi}} \frac{\sin \pi \mu \left(n + \frac{1}{2}\right) + o(1)}{\ln n} \frac{\Gamma\left(\mu \left(n + \frac{1}{2}\right)\right)}{\left(\mu \left(n + \frac{1}{2}\right)\right)^{n+1}}.$$

Replacing $\Gamma(\mu(n + \frac{1}{2}))$ by Stirling's formula and simplifying we finally obtain:

$$a_n = \frac{\sin \pi \mu \left(n + \frac{1}{2}\right) + o(1)}{\sqrt{\ln n}} \left[\mu \left(n + \frac{1}{2}\right)\right]^{\mu \left(n + \frac{1}{2}\right) - n} e^{-\mu \left(n + \frac{1}{2}\right)}.$$

5. The Phragmen-Lindelöf principle and the growth of an entire function in different directions

Just as the function $n(r)$ proved to be an insufficient characterization of the distribution of the zeros of an entire function $F(z)$, so too the function $\ln M(r)$ is an insufficient characterization of the behavior of $F(z)$ at infinity. Even in the most general questions of the theory of entire functions it is necessary to characterize the growth of the function in different directions. However, in introducing such characterizations it is necessary to understand clearly to what extent the growth of the function along a curve is independent of its growth along nearby curves.

The simplest result of this sort is the following theorem.

THEOREM 1. *If the function $F(z)$ is regular inside and on the sides of a sector of opening not exceeding π/ρ ($\rho \geq \frac{1}{2}$), is bounded on its sides, and $\ln |F(z)| = o(r^\alpha)$, $\alpha > 0$, as $r \rightarrow \infty$ uniformly in φ ($z = re^{i\varphi}$) inside this sector, then $F(z)$ is bounded throughout the sector.*

Proof: Without loss of generality we may assume that our sector is given by $|\arg z| < \pi/2\rho$. Consider the auxiliary function

$$F_{\epsilon}(z) = F(z) e^{-\epsilon z^{\rho-\alpha}}.$$

For sufficiently large r we have:

$$|F_{\epsilon}(re^{i\varphi})| = |F(re^{i\varphi})| e^{-\epsilon r^{\rho-\alpha} \cos(\rho-\alpha)\varphi} \leq e^{\rho(r^{\rho-\alpha}) - \epsilon r^{\rho-\alpha} \cos(\rho-\alpha)\varphi}.$$

Since for all φ inside our sector $\cos(\rho-\alpha)\varphi < \sin \pi\alpha/2\rho > 0$, r can be chosen so large that on the arc of the circle $|z|=r$ lying inside our sector the inequality

$$|F_{\epsilon}(z)| < M, \quad M = \sup_{|\arg z| = \frac{\pi}{2\rho}} |F(z)|.$$

is valid. On the sides of the sector this inequality is obviously satisfied, since

$$|\arg z| = \frac{\pi}{2\rho} \left| e^{-\epsilon z^{\rho-\alpha}} \right| \leq 1.$$

By the maximum principle this inequality holds throughout the interior of our sector.

But ϵ was arbitrary, and M does not depend on ϵ . Therefore we can pass to the limit $\epsilon \rightarrow 0$, and we obtain $|F(z)| \leq M$ inside the sector, which was to be proven.

We shall now improve this theorem.

THEOREM 2. *If $F(z)$ is regular inside and on the boundary of a sector of opening not exceeding π/ρ , $\rho \geq \frac{1}{2}$, bounded on the sides of the sector, and if $\ln |F(z)| = o(r^{\rho})$ as $r \rightarrow \infty$ uniformly in φ inside the sector, then $F(z)$ is bounded in the sector.*

Proof: Let us assume again that our sector is given by $|\arg z| < \pi/2\rho$, and let us consider the auxiliary function $F_{\epsilon}(z) = F(z) \exp(-\epsilon z^{\rho})$. On the sides of the sector

$$|F_\epsilon(z)| \leq M, \quad M = \sup_{|\arg z| = \frac{\pi}{p}} |F(z)|,$$

since for $|\arg z| = \pi/2\rho$, $|\exp(-\epsilon z^\rho)| = 1$. $F_\epsilon(z)$ is bounded by some constant M_ϵ on the real axis, since $|F_\epsilon(x)| = \exp(o(x^\rho) - \epsilon x^\rho)$.

Since $|F_\epsilon(z)| = \exp(o(|z|^\rho)) = \exp(o(|z|^{2\rho-\alpha}))$, we can apply Theorem 1 to $F_\epsilon(z)$ in the sectors $0 \leq \arg z \leq \pi/2\rho$ and $0 \geq \arg z \geq -\pi/2\rho$. Thus we find that $F_\epsilon(z)$ is bounded in the full sector $|\arg z| \geq \pi/2\rho$ by the larger of the two constants M and M_ϵ . But on the sides of this sector $|F_\epsilon(z)| \leq M$, and so by Theorem 1 again $|F_\epsilon(z)| \leq M$ throughout the angle $|\arg z| \leq \pi/2\rho$. Passing to the limit $\epsilon \rightarrow 0$ we obtain the assertion of the theorem.

Since the proof of Theorem 2 depended on the function $\exp(\epsilon z^\rho)$ that was bounded on the sides of the sector and tended to infinity inside the sector, it is natural to suppose that the generalization of Theorem 2 to an arbitrary domain will depend of the construction of a similar function. And the construction of such a function is quite easy. Let D be a simply connected domain having infinity as a boundary point, and let $w = \Psi(z)$ be the function mapping D conformally onto the half-plane $\operatorname{Re} w > 0$ and taking infinity into itself. Then $\exp(\Psi(z))$ will have the properties we want. The difficulty in the generalization arises from the fact that it is not clear what will correspond under this mapping to the curves $|w| = r$. This difficulty disappears when the mapping function is simple, or when this problem can be solved qualitatively. For example, if D is the half-strip $|\operatorname{Im} z| < \alpha$, $\operatorname{Re} z > 0$, then the mapping function is $\exp(\pi z/2\alpha)$, and the circles $|w| = r$ correspond to the lines $\operatorname{Re} z = x$. The corresponding theorem is as follows.

THEOREM 3. *If $F(z)$ is regular inside and on the boundary of a half-strip of width $2a$, is bounded on the sides, and if $|F(z)| = \exp(o(e^{|z|^{1/2}}/2a))$, then $F(z)$ is bounded inside this half-strip.*

If, for example, D is almost a half strip, that is, is bounded by curves $z = x + \frac{1}{2}ia(x)$, where $a'(x)$ and $xa'(x)/a(x)$ are monotonic functions for $x > x_0$, and tend to zero, then one can reason as follows. The function $w = \varphi(z)$, mapping D on the half-strip satisfies, for sufficiently large x , the relation $\operatorname{Im} \varphi(x + \frac{1}{2}ia(x)) = \frac{1}{2}\pi$. But

$$\operatorname{Im} \varphi\left(x + \frac{ia(x)}{2}\right) = \frac{a(x)}{2} \varphi'(x) - \left[\frac{a(x)}{2}\right]^3 \frac{\varphi'''(x)}{3!} + \dots$$

Therefore we have the relation

$$\pi = a(x) \varphi'(x) - \frac{1}{4} \left[a(x) \right]^3 \frac{\varphi'''(x)}{3!} + \dots,$$

from which, by "primitive" methods, it is easy to obtain the asymptotic behaviour of $\varphi'(x)$:

$$\varphi'(x) = \frac{\pi}{a(x)} + o(x^{-2+\epsilon}),$$

$$\varphi(x) = \pi \int_0^x \frac{dt}{a(t)} + C + o(x^{-1+\epsilon}).$$

This means that the limiting growth that enters into the statement of the Phragmen-Lindelof theorem is

$$e^{\pi \int_0^x \frac{dt}{a(t)}}. \quad (1)$$

We can obtain a similar estimate for domains that are close to a sector. In this case the limiting growth is

$$e^{\pi \int_0^{\ln x} \frac{dt}{\alpha(t)}} \quad (2)$$

Here $\alpha(t)$ is the angle subtended by the arc of the circle $|z|=t$ lying in D , from the point $z=0$.

The Phragmen-Lindelof principle is usually said to be the extension of the maximum principle to the case of an infinite domain. This is, however, merely a formal description which does not express the essence of the matter. The essential feature of the Phragmen-Lindelof principle is that it establishes a quantitative connection between the width of a domain in which a function is analytic and tends to infinity, and the order with which it tends to infinity. Formulae (1) and (2) do this for domains symmetric about the real axis, whose widths vary in a regular manner. Crudely speaking this connection says that the greater the growth of the function, the narrower is the domain in which it is possible.

The question to what extent turns in the domain affect the growth of a function is very difficult, and here we can say almost nothing about it.

The Phragmen-Lindelof principle is widely and effectively used in the theory of entire functions. We shall only consider its simplest application to questions of the growth of an entire function in different directions.

The basic characteristic of the growth of an entire function of finite order in different directions is the *indicator function* $H(\varphi)$, defined by the equation

$$H(\varphi) = \overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)},$$

where $h(r)$ is the order of growth of $F(z)$. (Note that

$h(z) = z^{\rho} l(z)$ and, consequently, $h(z)$ is regular for $|\arg z| < \pi/\rho - \delta$, $\delta > 0$.)

We shall establish several important properties of the indicator with the aid of Theorem 1.

1. 1. If $0 < \varphi_2 - \varphi_1 < \pi/\rho$, and c is a number such that

$$H(\varphi_1) = \operatorname{Re} c e^{i\rho\varphi_1}, \quad H(\varphi_2) = \operatorname{Re} c e^{i\rho\varphi_2} \quad \left(\rho = \lim_{r \rightarrow \infty} \frac{r h'(r)}{h(r)} \right),$$

then

$$H(\varphi) \leq \operatorname{Re} c e^{i\rho\varphi}, \quad \varphi_1 \leq \varphi \leq \varphi_2.$$

Proof: Choose the number c so that

$$\operatorname{Re} c e^{i\rho\varphi_1} = H(\varphi_1) + \varepsilon, \quad \operatorname{Re} c e^{i\rho\varphi_2} = H(\varphi_2) + \varepsilon,$$

and consider the function

$$F_{\varepsilon}(z) = F(z) e^{-c e^{i\rho\varphi_1} z}$$

in the angle $\varphi_1 \leq \arg z \leq \varphi_2$. (Without loss of generality we may assume that the negative real axis is not included in this angle.) Inside the angle

$$F_{\varepsilon}(z) = e^{\sigma(h(r))} = e^{\sigma(r^{\rho+\varepsilon})}$$

for all $\varepsilon < 0$. On the boundary $F_{\varepsilon}(z)$ is bounded, since

$$\operatorname{Re} c e^{i\rho\varphi_k} (r e^{i\varphi_k}) \sim r^{\rho} l(r) \operatorname{Re} c e^{i\rho\varphi_k} = h(r) (H(\varphi_k) + \varepsilon),$$

$k = 1, 2,$

and

$$|F(r e^{i\varphi_k})| < M_{\varepsilon} e^{h(r) (H(\varphi_k) + \varepsilon)}.$$

Since the opening of the angle is less than π/ρ we can

apply Theorem 1, from which it follows that $F_\epsilon(z)$ is bounded throughout the angle. This gives

$$|F(z)| < M e^{\operatorname{Re} \sigma_\epsilon h(z)}$$

or $H(\varphi) \leq \operatorname{Re} c_\epsilon e^{i\rho\varphi}$. Passing to the limit as $\epsilon \rightarrow 0$ we obtain the theorem.

Property 1 can be expressed in several different forms. For example, if we find c from the equations $H(\varphi_1) = \operatorname{Re} c e^{i\rho\varphi_1}$, $H(\varphi_2) = \operatorname{Re} c e^{i\rho\varphi_2}$, and substitute into $\operatorname{Re} c e^{i\rho\varphi}$, we obtain:

$$H(\varphi) \leq \frac{H(\varphi_1) \sin \rho(\varphi_2 - \varphi) + H(\varphi_2) \sin \rho(\varphi - \varphi_1)}{\sin \rho(\varphi_2 - \varphi_1)}, \quad \left. \begin{array}{l} \varphi_1 \leq \varphi \leq \varphi_2 < \varphi_1 + \frac{\pi}{\rho} \end{array} \right\} \quad (3)$$

From this we obtain

$$H(\varphi) \geq \frac{H(\varphi_1) \sin \rho(\varphi_2 - \varphi) + H(\varphi_2) \sin \rho(\varphi - \varphi_1)}{\sin \rho(\varphi_2 - \varphi_1)}, \quad \left. \begin{array}{l} \left\{ \begin{array}{l} \varphi_1 < \varphi_2 \leq \varphi < \varphi_1 + \frac{\pi}{\rho} \\ \varphi \leq \varphi_1 < \varphi_2 < \varphi + \frac{\pi}{\rho} \end{array} \right\} \end{array} \right\} \quad (4)$$

Equation (3) can be put into the symmetric form

$$\left. \begin{array}{l} H(\varphi_1) \sin \rho(\varphi_3 - \varphi_2) + H(\varphi_2) \sin \rho(\varphi_1 - \varphi_3) + \\ + H(\varphi_3) \sin \rho(\varphi_2 - \varphi_1) \geq 0, \quad \varphi_1 < \varphi_2 < \varphi_3 < \varphi_1 + \frac{\pi}{\rho} \end{array} \right\} \quad (5)$$

Property 1 is called *trigonometric ρ -convexity*.

2. If $H(\varphi) = -\infty$ for even one value of φ , then $H(\varphi) \equiv -\infty$.

The proof of this property follows at once from (3), since if $H(\varphi_1) = -\infty$, then $H(\varphi) = -\infty$ for $\varphi_1 \leq \varphi \leq \varphi_1 + \pi/\rho$. Applying this inequality to $\varphi_1 + \pi/\rho$, and then to $\varphi_1 + \pi/2\rho$, etc., we obtain $H(\varphi) \equiv -\infty$ for all φ .

Since

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \leq \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} < \infty,$$

property 2 says that $H(\varphi)$ is either finite everywhere, or $H(\varphi) \equiv -\infty$ (below we shall show that the second possibility can only occur when $F(z) \equiv 0$).

3. $H(\varphi)$ is continuous.

Proof. Choose an interval (φ_1, φ_2) , $0 < \varphi_2 - \varphi_1 < \pi/\rho$, and inside it a point φ_0 . Let $H_1(\varphi)$ and $H_2(\varphi)$ be the functions

$$H_1(\varphi) = \frac{H(\varphi_1) \sin \rho(\varphi_0 - \varphi) + H(\varphi_0) \sin \rho(\varphi - \varphi_1)}{\sin \rho(\varphi_0 - \varphi_1)},$$

$$H_2(\varphi) = \frac{H(\varphi_0) \sin \rho(\varphi_2 - \varphi) + H(\varphi_2) \sin \rho(\varphi - \varphi_0)}{\sin \rho(\varphi_2 - \varphi_0)}.$$

From inequalities (3) and (4) it follows that

$$H_2(\varphi) \leq H(\varphi) \leq H_1(\varphi), \quad \varphi_1 \leq \varphi \leq \varphi_0,$$

$$H_1(\varphi) \leq H(\varphi) \leq H_2(\varphi), \quad \varphi_0 \leq \varphi \leq \varphi_2.$$

Since $H_1(\varphi_0) = H_2(\varphi_0) = H(\varphi_0)$, we have:

$$H_2(\varphi) - H_2(\varphi_0) \leq H(\varphi) - H(\varphi_0) \leq H_1(\varphi) - H_1(\varphi_0), \quad \varphi_1 \leq \varphi \leq \varphi_0,$$

$$H_1(\varphi) - H_1(\varphi_0) \leq H(\varphi) - H(\varphi_0) \leq H_2(\varphi) - H_2(\varphi_0), \quad \varphi_0 \leq \varphi \leq \varphi_2.$$

The continuity of $H(\varphi)$ now follows from that of $H_1(\varphi)$ and $H_2(\varphi)$. In addition, $H(\varphi)$ satisfies a Lipschitz condition.

$$4. \quad \max_{0 \leq \varphi \leq 2\pi} H(\varphi) = \overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)}.$$

Proof: In the proof of property 1 we saw that for $0 < \varphi_2 - \varphi_1 < \pi/\rho$, $\varphi_1 < \varphi < \varphi_2$,

$$\ln |F(re^{i\varphi})| \leq \ln M_\bullet + h(r) \operatorname{Re} c_\bullet e^{i\rho\varphi},$$

where

$$\operatorname{Re} c_\bullet e^{i\rho\varphi} = \frac{(H(\varphi_1) + \varepsilon) \sin \rho(\varphi_2 - \varphi) + (H(\varphi_2) + \varepsilon) \sin \rho(\varphi - \varphi_1)}{\sin \rho(\varphi_2 - \varphi_1)}.$$

But $H(\varphi)$ is continuous; therefore $H(\varphi_1) = H(\varphi) + \varepsilon_1$, $H(\varphi_2) = H(\varphi) + \varepsilon_2$, where ε_1 and ε_2 tend to zero as $\varphi_1 \rightarrow \varphi$, $\varphi_2 \rightarrow \varphi$. From this we find:

$$\begin{aligned} & \leq \frac{(1 - \cos \rho(\varphi - \varphi_1) + \varepsilon + \varepsilon_1) \sin \rho(\varphi_2 - \varphi)}{\sin \rho(\varphi_2 - \varphi_1)} + \\ & + \frac{(1 - \cos \rho(\varphi_2 - \varphi) + \varepsilon + \varepsilon_2) \sin \rho(\varphi - \varphi_1)}{\sin \rho(\varphi_2 - \varphi_1)} \leq \\ & \leq 2\varepsilon + \varepsilon_1 + \varepsilon_2 + 2 - \cos \rho(\varphi - \varphi_1) - \cos \rho(\varphi_2 - \varphi) = \delta(\varphi_1, \varphi_2), \end{aligned}$$

where $\delta(\varphi_1, \varphi_2) \rightarrow 0$ for $\varphi_1 \rightarrow \varphi_2$. Consequently,

$$\frac{\ln |F(re^{i\varphi})|}{h(r)} \leq H(\varphi) + \delta(\varphi_1, \varphi_2), \quad \varphi_1 \leq \varphi \leq \varphi_2.$$

In particular, if the maximum of $\ln |F(re^{i\varphi})|$ is attained for $\varphi = \varphi(r)$, and if φ_0 is a limit point of $\varphi(r)$ as $r \rightarrow \infty$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} \leq H(\varphi_0) + \delta(\varphi_1, \varphi_2), \quad \varphi_1 < \varphi_0 < \varphi_2,$$

where δ is arbitrarily small, since φ_1 and φ_2 can be chosen arbitrarily close to φ_0 . Consequently,

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{h(r)} \leq H(\varphi_0) \leq \max_{\varphi_0 \leq \varphi \leq 2\pi} H(\varphi).$$

Since the converse inequality is obvious, property 4 is established.

These properties of the indicator show that the growth of an entire function depends continuously, in a certain sense, on the direction. The degree of continuity is determined by the Phragmen-Lindelöf principle in relation to the greatest growth of the function.

As an example of a similar result for functions of infinite order we present the following:

Let

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln \ln M(r)}{r} < \infty, \quad h(y) = \overline{\lim}_{x \rightarrow \infty} \frac{\ln \ln |F(x + iy)|}{x}.$$

Then the function $h(y)$ is continuous.

Closely connected with the Phragmen-Lindelöf principle is the following theorem, which connects the order of decrease of an entire function with its rate of increase.

THEOREM 4. Let $F(z)$ be an entire function. To each $\epsilon > 0$ there corresponds a sequence $r \rightarrow \infty$, such that $r_{n+1} - r_n < \epsilon r_n$ and

$$\min_{|z|=r_n} |F(z)| \geq \left\{ M\left(\frac{r_n}{1-\epsilon}\right) \right\}^{k(\epsilon)},$$

where $M(r)$ is the maximum modulus of $F(z)$ on $|z|=r$.

Proof: Without loss of generality we may assume that $F(0) \neq 0$. We shall prove the theorem by contradiction. If our assertion were false, then there would exist a sequence $r_n \rightarrow \infty$ such that

$$\min_{|z|=r} \ln |F(z)| < -k_n \ln M(r_n), \quad r_n(1-\epsilon) \leq r \leq r_n, \quad k_n \rightarrow \infty.$$

Let $z(r)$ denote the points where the minimum is attained, let E_n be the set of points $z(r)$ for $1-\epsilon \leq r/r_n \leq 1$, and let D_n be the domain obtained by removing the set E_n from the circle $|z| < r_n$. Let $T_1(z)$ be a harmonic function in D_n , equal to zero on $|z|=r_n$ and one for $z \in E_n$, and let

$T_1(z)$ be harmonic in D_n , equal to one on $|z|=r_n$ and to zero for $z \in E_n$. The logarithm of the modulus of an analytic function in a domain is less than or equal to a harmonic function, if it is less than or equal to the harmonic function on the boundary.

But on the boundary of D_n

$$\ln |F(z)| \leq \gamma_2(z) \ln M(r_n) - k_n \ln M(r_n) \gamma_1(z),$$

since for $|z|=r_n$ we have $\ln |F(z)| \leq \ln M(r_n)$, and for $z \in E_n$ $\ln |F(z)| \leq -k_n \ln M(r_n)$. Consequently

$$\ln |F(0)| \leq \gamma_2(0) \ln M(r_n) - k_n \gamma_1(0) \ln M(r_n).$$

Now note that, first of all, $T_2(0) < 0$ ($T_2(z) < 1$ for all z ($|z| < r_n$)), and, secondly, $T_1(0)$ does not depend on the position of the point $z(r) \in E_n$ on the circle $|z|=r$. Consequently, $T_1(0)$ is not less than $T(0)$, where $T(z)$ is the harmonic function in the circle $|z| < r_n$ with the slit $((1-\epsilon)r_n, r_n)$ on the real axis, equal to zero on $|z|=r_n$ and equal to one on the slit. Also, $T(0) > 0$ and is independent of n , by similarity considerations. This means that

$$\ln |F(0)| < (1 - k_n \gamma(0)) \ln M(r_n).$$

Since for $n \rightarrow \infty$ the right side of the inequality tends to $-\infty$, we reach a contradiction to the condition $F(0) \neq 0$. This proves the theorem.

Theorem 4, just as the Phragmen-Lindelöf principle, finds wide application in the theory of entire functions. Its meaning is that, crudely speaking, if we omit the zeros of $F(z)$ from consideration, then the order of decrease of $F(z)$ is $1/M(r)$ to some finite power. We shall consider only one example of the use of this theorem.

THEOREM 5. *Every entire function $F(z)$ of finite order can be represented in the form*

$$F(z) = F_1(z) e^{P(z)},$$

where $F_1(z)$ is the canonical product formed from the zeros of $F(z)$, and $P(z)$ is a polynomial whose degree is not greater than the order of $F(z)$.

Proof: In Section 3 we saw that $M_1(r) = \max_{|z| \leq r} |F_1(z)|$ had the growth $\exp(o(r^{\rho+1}))$, where ρ was the genus of $F_1(z)$. By Theorem 4 this means that on the sequence of circles $|z| = r_n$, $1 \leq r_{n+1}/r_n \leq 1 + \epsilon$,

$$|F_1(z)| > e^{o(r_n^{\rho+1})}.$$

Since by hypothesis $F(z) = \exp(O(r^{\rho+\epsilon}))$, $\rho < \infty$, we obtain:

$$\max_{|z|=r_n} |e^{P(z)}| \leq \max_{|z|=r_n} \frac{|F(z)|}{|F_1(z)|} = e^{O(r_n^{\rho+\epsilon})}.$$

This means that $\operatorname{Re} P(z) = O(r^{\rho+\epsilon})$, and by Liouville's theorem for harmonic functions, $P(z)$ is a polynomial of degree not exceeding ρ . The theorem is proven.

6. The connection between the indicator function and the distribution of zeros

The indicator function enables us to express more precisely the connection between the growth of an entire function and the distribution of its zeros. It turns out that the function $n(r)$, counting the number of zeros in a circle, is insufficient for our purposes; one must also know the distribution of zeros in different directions. For this purpose we introduce the function $n(r; \alpha, \beta)$, equal to the number of zeros of $F(z)$ in the sector $\alpha < \arg z < \beta$, $|z| < r$.

We begin with the simplest case, when $F(z)$ is an entire function of order less than one. In this case by Theorem 5 Section 5 it can be represented by a canonical product.

Consider the problem of estimating $|F(z)|$ from above and below, if all zeros of $F(z)$ lie in the sector $|\arg z - \varphi_0| < \eta$. As usual, $n(r)$ denotes the number of zeros in the circle $|z| < r$.

Just as in Section 3 we write:

$$\begin{aligned} \sum_{n=0}^{\infty} \ln \left| 1 - \frac{r}{|z_n|} e^{i(\varphi - \theta_n)} \right| &\leq \ln |F(re^{i\varphi})| = \\ &= \sum_{n=0}^{\infty} \ln \left| 1 - \frac{re^{i(\varphi - \theta_n)}}{z_n} \right| \leq \sum_{n=0}^{\infty} \ln \left| 1 - \frac{r}{|z_n|} e^{i(\varphi - \theta_n)} \right|, \end{aligned}$$

or

$$\begin{aligned} \int_0^{\infty} \ln \left| 1 - \frac{r}{t} e^{i(\varphi - \theta)} \right| dn(t) &\leq \ln |F(re^{i\varphi})| \leq \\ &\leq \int_0^{\infty} \ln \left| 1 - \frac{r}{\theta} \frac{t}{t} e^{i(\varphi - \theta)} \right| dn(t), \quad (1) \end{aligned}$$

where Θ_1 and Θ_2 depend only on φ and are defined as follows:

$$\begin{aligned} \theta_1(\varphi) &= \begin{cases} \varphi - \varphi_0 + \eta & \text{при } 0 \leq \varphi - \varphi_0 \leq \pi - \eta, \\ \pi & \text{при } \pi - \eta \leq \varphi - \varphi_0 \leq \pi, \end{cases} & \theta_1(\varphi - \varphi_0) &= \theta_1(\varphi_0 - \varphi), \\ \theta_2(\varphi) &= \begin{cases} 0 & \text{при } 0 \leq \varphi - \varphi_0 \leq \eta, \\ \varphi - \varphi_0 + \eta & \text{при } \eta \leq \varphi - \varphi_0 \leq \pi, \end{cases} & \theta_2(\varphi - \varphi_0) &= \theta_2(\varphi_0 - \varphi). \end{aligned}$$

It is easy to pass from inequality (1) to the indicator function. We note that if $n(r) \sim \Delta h(r)$, $h(r) = r^\rho l(r)$, $\rho < 1$, then for arbitrary $\varphi \neq 0$

$$\int_0^{\infty} \ln \left| 1 - \frac{re^{i\varphi}}{t} \right| dn(t) \sim \Delta \int_0^{\infty} \ln \left| 1 - \frac{re^{i\varphi}}{t} \right| dh(t),$$

and this last integral we have already estimated more than once in Sections 3 and 4. Therefore we have:

$$\lim_{r \rightarrow \infty} \frac{1}{h(r)} \int_0^{\infty} \ln \left| 1 - \frac{re^{i\varphi}}{t} \right| dn(t) = \frac{\pi \Delta}{\sin \pi \rho} \cos \rho (\pi - \varphi),$$

$$0 \leq \varphi \leq \pi,$$

and for $\varphi=0, 2\pi$ we have by the continuity of the indicator function

$$\overline{\lim}_{r \rightarrow \infty} \frac{1}{h(r)} \int_0^{\infty} \ln \left| 1 - \frac{re^{i\varphi}}{t} \right| dn(t) = \frac{\pi \Delta}{\sin \pi \rho} \cos \pi \rho.$$

Thus, if we divide both sides of (1) by $h(r)$ and pass to the limit we obtain:

If $n(r) \sim \Delta h(r)$, $h(r) = r^{\rho} l(r)$, $0 < \rho < 1$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \leq \begin{cases} \frac{\pi \Delta}{\sin \pi \rho} \cos \rho (\pi - |\varphi - \varphi_0| - \eta), & |\varphi - \varphi_0| \leq \pi - \eta, \\ \frac{\pi \Delta}{\sin \pi \rho}, & \pi - \eta \leq |\varphi - \varphi_0| \leq \pi. \end{cases}$$

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \geq \frac{\pi \Delta}{\sin \pi \rho} \cos \rho (\pi - |\varphi - \varphi_0| + \eta), \quad \eta \leq |\varphi - \varphi_0| \leq \pi,$$

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \geq \frac{\pi \Delta}{\sin \pi \rho} \cos \pi \rho, \quad 0 \leq |\varphi - \varphi_0| \leq \eta.$$

If we introduce the notations

$$H_{\eta}^{+}(\varphi) = \begin{cases} \frac{\pi}{\sin \pi \rho} \cos \rho (\pi - |\varphi| - \eta), & |\varphi| \leq \pi - \eta, \\ \frac{\pi}{\sin \pi \rho}, & \pi - \eta \leq |\varphi| \leq \pi, \end{cases}$$

$$H_{\eta}^{-}(\varphi) = \begin{cases} \frac{\pi}{\sin \pi \rho} \cos \pi \rho, & |\varphi| \leq \eta, \\ \frac{\pi}{\sin \pi \rho} \cos \rho (\pi - |\varphi| + \eta), & \eta \leq |\varphi| \leq \pi, \end{cases}$$

then these relations can be written in the form

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \leq \Delta H_{\eta}^{+}(\varphi - \varphi_0), \quad |\varphi - \varphi_0| \leq \pi,$$

$$\lim_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \geq \Delta H_{\eta}^{-}(\varphi - \varphi_0), \quad \eta \leq |\varphi - \varphi_0| \leq \pi, \quad (2)$$

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \geq \Delta H_{\eta}^{-}(\varphi - \varphi_0), \quad |\varphi - \varphi_0| \leq \pi.$$

In particular, the first and third formulae give:

$$\Delta H_{\eta}^{-}(\varphi - \varphi_0) \leq H(\varphi) \leq \Delta H_{\eta}^{+}(\varphi - \varphi_0).$$

It is not difficult to see that as $\eta \rightarrow 0$, $H_{\eta}^{+}(\varphi)$ and $H_{\eta}^{-}(\varphi)$ tend to $H_0(\varphi) = [\pi \cos \rho (\pi - |\varphi|)] / \sin \pi \rho$, $|\varphi| \leq \pi$.

With the aid of (2) we can easily obtain the result we want.

To simplify the formulation of the theorem we introduce the notation $n(r, \varphi) = n(r; 0, \varphi)$ —the number of zeros of $F(z)$ in the sector $0 < \arg z < \varphi$, $|z| < r$.

THEOREM 1. *If the limit*

$$\nu(\varphi) = \lim_{r \rightarrow \infty} \frac{n(r, \varphi)}{h(r)},$$

exists, then

$$H(\varphi) = \lim_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} = \int_0^{2\pi} H_0(\varphi - \theta) d\nu(\theta),$$

$$H_0(\varphi) = \frac{\pi \cos \rho(\pi - |\varphi|)}{\sin \pi \rho}, \quad |\varphi| \leq \pi. \quad (3)$$

Proof: Write $F(z)$ as the product

$$F(z) = F_1(z) F_2(z) \dots F_n(z),$$

where $F_k(z)$ is the canonical product formed from the zeros of $F(z)$ lying in the sector $S_k: |\arg z - 2\pi k/n| < \pi/n$. (Without loss of generality we may assume that $F(z)$ has no zeros on the sides of these sectors.) By (2)

$$\Delta_k H_{\frac{\pi}{n}}^-\left(\varphi - \frac{2k\pi}{n}\right) \leq H_k(\varphi) \leq \Delta_k H_{\frac{\pi}{n}}^+\left(\varphi - \frac{2k\pi}{n}\right), \quad |\varphi| \leq \pi,$$

$$\lim_{r \rightarrow \infty} \frac{\ln |F_k(re^{i\varphi})|}{h(r)} \geq \Delta_k H_{\frac{\pi}{n}}^-\left(\varphi - \frac{2k\pi}{n}\right), \quad \frac{\pi}{n} \leq \left|\varphi - \frac{2k\pi}{n}\right| \leq \pi,$$

$$\Delta_k = \nu\left(\frac{(2k+1)\pi}{n}\right) - \nu\left(\frac{(2k-1)\pi}{n}\right), \quad H_k(\varphi) = \overline{\lim}_{n \rightarrow \infty} \frac{\ln |F_k(re^{i\varphi})|}{h(r)}.$$

Consequently

$$H(\varphi) = \overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{h(r)} \leq \sum_{k=1}^n H_k(\varphi) \leq$$

$$\leq \sum_{k=1}^n H_{\frac{\pi}{n}}^+\left(\varphi - \frac{2k\pi}{n}\right) \left\{ \nu\left(\frac{(2k+1)\pi}{n}\right) - \nu\left(\frac{(2k-1)\pi}{n}\right) \right\},$$

$$H(\varphi) \geq \overline{\lim}_{r \rightarrow \infty} \frac{\ln |F_m(re^{i\varphi})|}{h(r)} + \sum_{k \neq m} \lim_{r \rightarrow \infty} \frac{\ln |F_k(re^{i\varphi})|}{h(r)} \geq$$

$$\geq \sum_{k=1}^n H_{\frac{\pi}{n}}^-\left(\varphi - \frac{2k\pi}{n}\right) \left\{ \nu\left(\frac{(2k+1)\pi}{n}\right) - \nu\left(\frac{(2k-1)\pi}{n}\right) \right\},$$

where m is determined by the inequality $|\varphi - 2\pi m/n| \leq \pi/n$.

Since $H_{\pi}^{+}(\varphi) \rightarrow H_0(\varphi)$, $H_{\pi}^{-}(\varphi) \rightarrow H_0(\varphi)$ as $n \rightarrow \infty$, we have

$$\sum_{k=1}^n H_{\frac{\pi}{n}}^{\pm} \left(\varphi - \frac{2k\pi}{n} \right) \left\{ \nu \left(\frac{2k+1}{n} \pi \right) - \nu \left(\frac{2k-1}{n} \pi \right) \right\} \sim \\ \sim \sum_{k=1}^n H_0 \left(\varphi - \frac{2k\pi}{n} \right) \left\{ \nu \left(\frac{2k+1}{n} \pi \right) - \nu \left(\frac{2k-1}{n} \pi \right) \right\}.$$

This last sum is the approximating sum to the Stieltjes integral

$$\int_0^{2\pi} H_0(\varphi - \nu) d\nu(\nu),$$

which proves the theorem. Formula (3) is valid for any ρ not an integer. At the same time inequality (1) becomes a little bit more complicated (see Section 2), however the Stieltjes integral is obtained by a passage to the limit in just the same way.

Formula (3) shows in a remarkably clear manner the connection between the distribution of zeros and the indicator function, in case the zeros are regularly distributed. A number of very interesting corollaries can be obtained from (3). Note, for example, that $H_0(\varphi)$ is a continuous function, and has a continuous derivative for all φ except $\varphi=0$, while at $\varphi=0$ the derivative has a jump of $2\pi\rho$. This means that $H(\varphi)$ has a continuous derivative at all points of continuity of $\nu(\varphi)$, while a jump of Δ in $\nu(\varphi)$ corresponds to a jump of $2\pi\rho\Delta$ in $H'(\varphi)$. But the jump of $\nu(\varphi)$ at $\varphi=\varphi_0$ is equal to the density of zeros of $F(z)$, lying in an arbitrarily small angle con-

taining the ray $\arg z = \varphi_0$. Thus, the jump of $H'(\varphi)$ at $\varphi = \varphi_0$ is equal to $2\pi\rho\Delta$, where Δ is the density of zeros of $F(z)$ in an arbitrary small angle containing the ray $\arg z = \varphi_0$.

Since the quantity $2\pi\rho\Delta$ depends continuously on ρ one might expect that our assertion will remain true for ρ an integer. This is indeed the case, although formula (3) is not true for ρ an integer. The analogous formula is a little more complicated. We shall only consider the simplest case $\rho = 1$, $p = 0$.

In Section 3, we saw that the order of growth of $F(z)$ for $\rho = 1$, $p = 0$ was given by the larger of the two numbers $n(r)$, $r\tau(r)$, where

$$\tau(r) = \left| \sum_{|z_n| > 2r} z_n^{-1} \right|.$$

The most important case is when $r\tau(r) = o(n(r))$. This is only possible when there is a lot of cancellation in the series $\sum 1/z_n$, that is, when there are about equal numbers of zeros with arguments differing by an amount close to π .

Just as before, we begin by solving the corresponding extremal problem. In this case it takes the following form:

Let

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

where z_n are the zeros of $F(z)$ lying in the angles $|\arg z - \varphi_0| < \eta$, $|\arg z - \varphi_0 - \pi| < \eta$, and $n_1(r)$ denotes the number of zeros in the first angle lying in the circle $|z| < r$, and $n_2(r)$ denotes the number in the second angle lying inside the circle. Also, let

$$\left| \sum_{|z_n| > r} z_n^{-1} \right| = \tau(r).$$

Find $\max \ln |F(re^{i\varphi})|$ and $\min \ln |F(re^{i\varphi})|$, where the maximum and minimum are taken over all z_n with fixed moduli (that is, the functions $n_1(r)$ and $n_2(r)$ are fixed), and with arguments lying in the corresponding angles.

Aside from the fact that these extremal problems are somewhat more complicated than those encountered before, their solution does not present any great difficulty.

We have:

$$\begin{aligned} \ln |F(re^{i\varphi})| &= \sum_{|z_n| < r} \ln \left| 1 - \frac{re^{i\varphi}}{z_n} \right| + \sum_{|z_n| > r} \ln \left| 1 - \frac{re^{i\varphi}}{z_n} \right| = \\ &= S_1(r, \varphi) + S_2(r, \varphi). \end{aligned}$$

Let us estimate $S_1(r, \varphi)$. Let z'_n denote zeros of $F(z)$ lying in the angle $|\arg z - \varphi_0| < \eta$, and let z''_n denote zeros of $F(z)$ lying in the angle $|\arg z - \varphi_0 - \pi| < \eta$. We obtain:

$$S_1(r, \varphi) = \sum_{|z'_n| < r} \ln \left| 1 - \frac{re^{i\varphi}}{z'_n} \right| + \sum_{|z''_n| < r} \ln \left| 1 - \frac{re^{i\varphi}}{z''_n} \right|.$$

But for fixed α , $\ln |1 - \alpha e^{i\theta}|$ is an increasing function of $|\theta|$, and therefore

$$\begin{aligned} S_1(r, \varphi) &\leq \int_0^r \ln \left| 1 - \frac{t}{t} e^{i\theta_1^+(\varphi - \varphi_0)} \right| dn_1(t) + \\ &\quad + \int_0^r \ln \left| 1 - \frac{t}{t} e^{i\theta_2(\varphi - \varphi_0 - \pi)} \right| dn_2(t), \quad (4) \end{aligned}$$

where

$$\theta_1^+(\varphi) = \begin{cases} |\varphi| + \eta & |\varphi| \leq \pi - \eta, \\ \pi & \pi - \eta \leq \varphi \leq \pi. \end{cases} \quad (5)$$

In exactly the same way

$$S_1(r, \varphi) \geq \int_0^{2r} \ln \left| 1 - \frac{r}{t} e^{i\theta_1^-(\varphi - \varphi_0)} \right| dn_1(t) + \\ + \int_0^{2r} \ln \left| 1 - \frac{r}{t} e^{i\theta_1^-(\varphi - \varphi_0 - \pi)} \right| dn_2(t), \quad (6)$$

where

$$\theta_1^-(\varphi) = \begin{cases} 0 & |\varphi| \leq \eta, \\ |\varphi| - \eta & \eta \leq |\varphi| \leq \pi. \end{cases} \quad (7)$$

It is a little more difficult to estimate $S_2(r, \varphi)$. We have:

$$S_2(r, \varphi) + \operatorname{Re} r e^{i\varphi} \sum_{|z_n| > 2r} z_n^{-1} = \\ = \sum_{|z_n| > 2r} \ln \left| 1 - \frac{r e^{i\varphi}}{z_n} \right| + \operatorname{Re} \frac{r e^{i\varphi}}{z_n} = \\ = -\operatorname{Re} \sum_{|z_n| > 2r} \left\{ \frac{r^2 e^{2i\varphi}}{2z_n^2} + \frac{r^3 e^{3i\varphi}}{3z_n^3} + \dots \right\}.$$

Let us consider the dependence of the quantity

$$\operatorname{Re} \sum_{m=2}^{\infty} \frac{\alpha^m e^{im\theta}}{m}$$

on θ . It is easy to see that it has two maxima at $\theta=0$ and $\theta=\pi$ and two minima, at $\cos \theta = \alpha/2$. Therefore

$$S_2(r, \varphi) + \operatorname{Re} r e^{i\varphi} \sum_{|z_n| > 2r} z_n^{-1} \leq \\ \leq \int_{2r}^{\infty} \sum_{m=2}^{\infty} \frac{r^m}{m t^m} \cos m \theta_2^+(\varphi - \varphi_0, t) dn_1(t) + \\ + \int_{2r}^{\infty} \sum_{m=2}^{\infty} \frac{r^m}{m t^m} \cos m \theta_2^+(\varphi - \varphi_0 - \pi, t) dn_2(t). \quad (8)$$

$$\begin{aligned}
& S_2(r, \varphi) + \operatorname{Re} r e^{i\varphi} \sum_{|z_n| > 2r} z_n^{-1} \geq \\
& \geq \int_{2r}^{\infty} \sum_{m=2}^{\infty} \frac{r^m}{m t^m} \cos m \theta_2^-(\varphi - \varphi_0, t) dn_1(t) + \\
& + \int_{2r}^{\infty} \sum_{m=2}^{\infty} \frac{r^m}{m t^m} \cos m \theta_2^-(\varphi - \varphi_0 - \pi, t) dn_2(t). \quad (9)
\end{aligned}$$

Here $\Theta_2^+(\varphi, t)$ and $\Theta_2^-(\varphi, t)$ are functions for which

$$|\theta_2^+(\varphi, t) - |\varphi|| \leq \eta, \quad |\theta_2^-(\varphi, t) - |\varphi|| \leq \eta.$$

Formulae (4), (6), (8) and (9) provide the solution to our extremal problem. Although this solution is not very effective, it does enable us to pass to the indicator function. The corresponding indicators $H_\eta^+(\varphi)$ and $H_\eta^-(\varphi)$ are not computable, however $H_\eta^+(\varphi) \rightarrow H_0(\varphi)$ and $H_\eta^-(\varphi) \rightarrow H_0(\varphi)$ as $\eta \rightarrow 0$, and it is quite easy to calculate $H_0(\varphi)$. Indeed, let us assume that $n_1(t) \sim n_2(t) \sim \Delta l(t)$, $\tau(t) = o(l(t))$. Then

$$\begin{aligned}
S_1^+(r, \varphi) & \sim \Delta \int_0^{2r} \ln \left| 1 - \frac{r}{t} e^{i\theta_1^+(\varphi - \varphi_0)} \right| l(t) dt + \\
& + \Delta \int_0^{2r} \ln \left| 1 - \frac{r}{t} e^{i\theta_1^+(\varphi - \varphi_0 - \pi)} \right| l(t) dt \sim \\
& \sim \Delta r l(r) \int_0^2 \ln \left| \left(1 - \frac{1}{u} e^{i\theta_1^+(\varphi - \varphi_0)} \right) \right. \\
& \quad \left. \left(1 - \frac{1}{u} e^{i\theta_1^+(\varphi - \varphi_0 - \pi)} \right) \right| du
\end{aligned}$$

and similarly

$$S_1^-(r, \varphi) \sim \Delta r l(r) \int_0^{\pi} \ln \left| \left(1 - \frac{e^{i\theta_1^-(\varphi - \varphi_0)}}{u} \right) \left(1 - \frac{e^{i\theta_1^-(\varphi - \varphi_0 - \pi)}}{u} \right) \right| du.$$

Since $|\Theta_1^+(\varphi) - |\varphi|| \leq \eta$, $|\Theta_1^-(\varphi) - |\varphi|| \leq \eta$, for $\eta \rightarrow 0$ we obtain:

$$\left. \begin{aligned} \lim_{r \rightarrow \infty} \frac{S_1^+(r, \varphi)}{r l(r)} &\sim \Delta \int_0^2 \ln \left| 1 - \frac{e^{2i|\varphi - \varphi_0|}}{u^2} \right| du, \quad |\varphi - \varphi_0| \leq \pi, \\ \lim_{r \rightarrow \infty} \frac{S_1^-(r, \varphi)}{r l(r)} &\sim \Delta \int_0^2 \ln \left| 1 - \frac{e^{2i|\varphi - \varphi_0|}}{u^2} \right| du, \\ \eta &\leq |\varphi - \varphi_0| \leq \pi - \eta, \\ \overline{\lim}_{r \rightarrow \infty} \frac{S_1^-(r, \varphi)}{r l(r)} &\sim \Delta \int_0^2 \ln \left| 1 - \frac{e^{2i|\varphi - \varphi_0|}}{u^2} \right| du, \quad |\varphi - \varphi_0| \leq \pi. \end{aligned} \right\}$$

For $S_2(r, \varphi)$ we obtain in the same manner

$$\left. \begin{aligned} \lim_{r \rightarrow \infty} \frac{S_2^+(r, \varphi)}{r l(r)} &\sim \Delta \int_2^{\infty} \ln \left| 1 - \frac{e^{2i(\varphi - \varphi_0)}}{u^2} \right| du, \quad |\varphi - \varphi_0| \leq \pi, \\ \lim_{r \rightarrow \infty} \frac{S_2^-(r, \varphi)}{r l(r)} &\sim \Delta \int_2^{\infty} \ln \left| 1 - \frac{e^{2i|\varphi - \varphi_0|}}{u^2} \right| du, \\ \eta &\leq |\varphi - \varphi_0| < \pi - \eta, \\ \overline{\lim}_{r \rightarrow \infty} \frac{S_2^-(r, \varphi)}{r l(r)} &\sim \Delta \int_2^{\infty} \ln \left| 1 - \frac{e^{2i|\varphi - \varphi_0|}}{u^2} \right| du, \quad |\varphi - \varphi_0| \leq \pi. \end{aligned} \right\}$$

Thus we come to the conclusion:

If $F(z)$ is a canonical product of genus zero and order one, whose zeros lie in the angles $|\arg z - \varphi_0| < \eta$ and $|\arg z - \varphi_0 - \pi| < \eta$, while $n_1(r)$ denotes the number of zeros in the first angle lying in the circle $|z| < r$, and $n_2(r)$ denotes the number of zeros in the second angle lying in this circle, and if $n_1(r) \sim n_2(r) \sim rl(r)$ and

$$\sum_{|z_n| > 2r} z_n^{-1} = o(l(r)),$$

then for small η

$$\left. \begin{aligned} \overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{rl(r)} &\leq \Delta H_{\eta}^{+}(\varphi - \varphi_0) \sim \Delta H_0(\varphi - \varphi_0), \\ &|\varphi - \varphi_0| \leq \pi, \\ \lim_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{rl(r)} &\geq \Delta H_{\eta}^{-}(\varphi - \varphi_0) \sim \Delta H_0(\varphi - \varphi_0), \\ &\eta < |\varphi - \varphi_0| < \pi - \eta, \\ \overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{rl(r)} &\geq \Delta H_{\eta}^{-}(\varphi - \varphi_0) \sim \Delta H_0(\varphi - \varphi_0), \\ &|\varphi - \varphi_0| \leq \pi. \end{aligned} \right\} (12)$$

Here

$$H_0(\varphi) = \int_0^{\infty} \ln \left| 1 - \frac{e^{2i|\varphi|}}{u^2} \right| du = \pi |\sin \varphi|.$$

Repeating almost word for word the reasoning used in proving Theorem 1 we come to the formula

$$H(\varphi) = \frac{1}{2} \int_0^{2\pi} H_0(\varphi - \theta) d\nu(\theta). \quad (13)$$

The factor $\frac{1}{2}$ comes from the fact that each ray has been counted twice. We proved this formula under the assumption that

$$\sum_{|z_n| > 2r} z_n^{-1} = o(l(r)).$$

If we were to replace this by the assumption that

$$\lim_{r \rightarrow \infty} \frac{1}{l(r)} \sum_{|z_n| > 2r} z_n^{-1} = A, \quad (14)$$

then we would obtain the formula

$$H(\varphi) = \frac{1}{2} \int_0^{2\pi} H_0(\varphi - \theta) d\nu(\theta) + \operatorname{Re} A e^{i\varphi}. \quad (15)$$

Condition (14) is the same sort of regularity assumption about the distribution of zeros as is the assumption of the existence of the limit

$$\lim_{r \rightarrow \infty} \frac{n(r, \varphi)}{h(r)} = \nu(\varphi).$$

The function $\tau(r)$ characterizes the asymmetry in the distribution of the zeros z_n , since it is clear that in the sum $\tau(r)$ symmetric zeros cancel each other. Consequently, the existence of the limit A implies a certain smoothness in the non-symmetric zeros.

From (15) it is clear, in particular, that zeros having density $n(r) = o(h(r))$ cannot cause a jump in the indicator function. We now say a few words about extending our results to all remaining cases.

It is clear that when the genus and order of $F(z)$ are both equal to one there is almost no change from the case we considered. All the estimates are repeated unchanged, except that we must define

$$\tau(r) = \sum_{|z_n| < 2r} z_n^{-1}.$$

instead of

$$\tau(r) = \sum_{|z_n| > 2r} z_n^{-1}$$

The case of arbitrary integral order is somewhat harder. Let us assume for simplicity that $\nu(\Theta + \pi/\rho) = \nu(\Theta)$. The basic extremal problem in this case is the problem of estimating the canonical product $F(z)$, having zeros in the sectors $|\arg z - \varphi_0| < \eta$ and $|\arg z - \varphi_0 - \pi/\rho| < \eta$, under the condition that

$$\left| \sum_{|z_n| > 2r} z_n^{-\rho} \right| \leq \tau(r) \text{ or } \left| \sum_{|z_n| < 2r} z_n^{-\rho} \right| \leq \tau(r).$$

If we assume that $\tau(r) = o(n(r)/r^\rho)$, then for the indicator $H_0(\varphi)$ we obtain the indicator of the function $F_\rho(z e^{i\pi/2\rho})$

$$F_\rho(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z e^{\frac{\pi i}{2\rho}}}{\frac{1}{n^\rho}} \right) \left(1 - \frac{z e^{-\frac{\pi i}{2\rho}}}{\frac{1}{n^\rho}} \right) \exp \sum_{m=1}^{\rho-1} \frac{z^m \cos \frac{m\pi}{2\rho}}{m n^{\frac{m}{\rho}}},$$

which we have already met at the end of Section 4.

With the help of this indicator the corresponding formula can be written in the form

$$H(\varphi) = \frac{1}{2} \int_0^{2\pi} H_0(\varphi - \theta) d\nu(\theta) + \operatorname{Re} A e^{i\varphi}.$$

Here

$$H_0(\varphi) = \begin{cases} \frac{\pi}{\rho} \sin \rho \varphi, & 0 \leq \varphi \leq \frac{\pi}{\rho}, \\ -\pi \frac{2\rho-1}{\rho} \sin \rho \varphi & \text{for all other } \varphi, \end{cases}$$

$$A = \lim_{r \rightarrow \infty} \frac{r^\rho}{h(r)} \sum_{|z_n| > 2r} z_n^{-\rho}.$$

The jump of $H_0(\varphi)$ at $\varphi=0$ is equal to $2\pi\rho$, so that our remark that the jump of the indicator function at $\varphi=\varphi_0$

is equal to $2\pi\rho$ multiplied by the density of the zeros in an arbitrarily small angle containing the ray $\arg z = \varphi_0$ remains valid for arbitrary ρ .⁶

7. The Borel transform

To conclude this chapter we present one other powerful method for studying the behavior of an entire function at infinity. This method consists in establishing a relation between the behavior of an entire function at infinity and the distribution and character of the singular points of a certain other function, regular outside a finite circle. This method is mainly applied to entire functions of exponential type, that is, to functions having the exact order of growth $h(r) = r$.

The method is based on the following integral transformation.

Let $F(z) = \sum a_n z^n / n!$ be an entire function of exponential type, that is,

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln M(r)}{r} < \infty. \quad (1)$$

The function $f(\zeta) = \sum a_n / \zeta^{n+1}$, which is regular at infinity by (1), is called the *Borel transform* of $F(z)$.

The functions $F(z)$ and $f(\zeta)$ are obtained from one another by the formulae

$$F(z) = \frac{1}{2\pi i} \int_C f(\zeta) e^{z\zeta} d\zeta, \quad (2)$$

⁶ Questions concerning the connection between the angular density of zeros and the indicator of an entire function are studied in detail in the book of B. Ya. Levin: *Distribution of the zeros of entire functions*, Moscow, 1956 (Russian).

$$f(\zeta) = \frac{1}{\zeta} \int_0^{\infty} F\left(\frac{x}{\zeta}\right) e^{-x} dx. \quad (3)$$

Here the contour C contains all the singular points of $f(\zeta)$.

These formulae are simple to prove. To prove (2) we move the contour C so that the series for $f(\zeta)$ converges uniformly on it. Then integrating term-by-term we obtain:

$$\frac{1}{2\pi l} \int_C f(\zeta) e^{z\zeta} d\zeta = \sum_{n=0}^{\infty} \frac{a_n}{2\pi l} \int_C \frac{e^{z\zeta}}{\zeta^{n+1}} d\zeta = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!} = F(z).$$

To prove (3) note that from (1) we have

$$\left| F\left(\frac{x}{\zeta}\right) \right| \leq M e^{A \frac{x}{|\zeta|}}.$$

Thus, if $|\zeta| > A$ then we can integrate termwise, and we obtain:

$$\frac{1}{\zeta} \int_0^{\infty} F\left(\frac{x}{\zeta}\right) e^{-x} dx = \sum_{n=0}^{\infty} \frac{a_n}{n!} \frac{1}{\zeta^{n+1}} \int_0^{\infty} x^n e^{-x} dx = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^{n+1}} = f(\zeta).$$

We have established this formula for $|\zeta| > A$, but by the principle of analytic continuation it is valid in any simply connected domain in which the integral converges uniformly, and which contains the point $\zeta = \infty$.

Formulae (2) and (3) enable us to establish connections between the properties of $f(\zeta)$ and $F(z)$. As the simplest example of such a connection we present the following theorem.

THEOREM 1. *Let E be the set of all singular points of $f(\zeta)$, and E_1 , the smallest convex set containing E , and let $H(\varphi)$ be the indicator function of $F(z)$, that is, $H(\varphi) = \limsup \ln |F(re^{i\varphi})|/r$. Then*

$$H(\varphi) = \sup_{\zeta \in E_i} \operatorname{Re} \zeta e^{i\varphi} = K(\varphi). \quad (4)$$

Proof: In formula (3) we can choose the contour C so that all points of it have distance less than ϵ from the set E . Then

$$\begin{aligned} |F(re^{i\varphi})| &\leq \frac{1}{2\pi} \int_C |f(\zeta)| e^{r \operatorname{Re} \zeta e^{i\varphi}} |d\zeta| \leq \\ &\leq M_\epsilon e^{r \max_{\zeta \in C} \operatorname{Re} \zeta e^{i\varphi}} \leq M_\epsilon e^{r(K(\varphi) + \epsilon)}. \end{aligned}$$

Since ϵ was arbitrary, we find that $H(\varphi) \leq K(\varphi)$.

On the other hand, in formula (3) let $\zeta = te^{-i\varphi}$.

Then we have:

$$f(\zeta) = \frac{1}{\zeta} \int_0^\infty F\left(\frac{x e^{i\varphi}}{t}\right) e^{-x} dx = e^{i\varphi} \int_0^\infty F(ue^{i\varphi}) e^{-ut} du.$$

These operations are certainly valid if t is positive and sufficiently large, which means that by analytic continuation the formula

$$f(\zeta) = e^{i\varphi} \int_0^\infty F(ue^{i\varphi}) e^{-u\zeta e^{-i\varphi}} du$$

is valid wherever the integral converges. Since

$$\lim_{u \rightarrow \infty} \frac{\ln |F(ue^{i\varphi})|}{u} = H(\varphi),$$

the integral converges uniformly for

$$\operatorname{Re} \zeta e^{i\varphi} > H(\varphi).$$

that is, $f(\zeta)$ is regular at each point ζ satisfying the condition

$$\operatorname{Re} \zeta e^{i\varphi} \geq H(\varphi) + \varepsilon,$$

This means that $H(\varphi) \geq \sup \operatorname{Re} \zeta e^{i\varphi}$ ($\zeta \in E$). But it is easy to see that

$$\sup_{\zeta \in E} \operatorname{Re} \zeta e^{i\varphi} = \sup_{\zeta \in E_1} \operatorname{Re} \zeta e^{i\varphi} = K(\varphi).$$

Consequently, $H(\varphi) = K(\varphi)$ and the theorem is proven.

The function $K(\varphi)$ has a simple geometric interpretation. Indeed, $\operatorname{Re} \zeta e^{i\varphi}$ is the length of the projection of the vector ζ on the ray $\arg \zeta = -\varphi$, and $\sup \operatorname{Re} \zeta e^{i\varphi}$ ($\zeta \in E_1$) is the largest value of such projections for all the points of E_1 . That is, $K(\varphi)$ is the length of the projection of the set E_1 on the ray $\arg \zeta = -\varphi$. The function $K(-\varphi)$ is called the *supporting function* of the set E_1 .

In view of Theorem 1 the properties of the indicator function become obvious geometrically.

If we have information about the character of the singular points of $f(\zeta)$, as well as about their distribution, then we can obtain corresponding properties of $F(z)$.

In many questions in the theory of entire functions of exponential type it is more convenient to use the singular points of $f(\zeta)$, rather than the indicator function, as the characteristic of the growth in different directions. The singular points are simpler to use and give a somewhat more accurate picture of the growth.

It is possible to construct an apparatus similar to the Borel transform for any entire function with an exact order of growth. (If $F(z)$ has order ρ equal to zero or infinity then, of course, both the indicator and the set of singular points are purely formal constructions with no real content.)

The principle of constructing the corresponding integral transforms is rather simple.

Let

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_n}, \quad m_n = \int_0^{\infty} x^n \mu(x) dx, \quad \mu(x) > 0.$$

The function $f(\zeta) = \sum a_n / \zeta^{n+1}$ is called the Φ -transform of the entire function $F(z) = \sum a_n z^n / m_n$.

The functions $f(\zeta)$ and $F(z)$ are expressed in terms of each other by the formulae

$$F(z) = \frac{1}{2\pi i} \oint_C f(\zeta) \Phi(z\zeta) d\zeta,$$

$$f(\zeta) = \frac{1}{\zeta} \int_0^{\infty} F\left(\frac{x}{\zeta}\right) \mu(x) dx$$

(the contour C contains all the singular points of $f(\zeta)$).

The properties of this generalized Borel transform are not as good, in general, as those of the ordinary transform. To understand this, let us consider one of the simplest examples:

$$\Phi(z) = E_{\rho}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\frac{n}{\rho} + 1\right)}, \quad \mu(x) = \rho x^{\rho-1} e^{-x^{\rho}}.$$

To formulate a theorem similar to Theorem 1 we need one more definition.

Let B be an arbitrary bounded point set in the ζ -plane. Let $D_{\rho}(\varphi, c)$ be the complementary domain with respect to the whole plane to the closed domain $\operatorname{Re} \zeta^{\rho} e^{i\rho\varphi} \geq c$, which contains part of the ray $\arg \zeta = -\varphi$. (For $\rho = 1$ $D_{\rho}(\varphi, c)$ is a half-plane, lying at distance c from $\zeta = 0$, with boundary perpendicular to the ray $\arg \zeta = -\varphi$.)

Let B_{ρ} denote the common part of all possible domains $D_{\rho}(\varphi, c)$ that contain the set B (ρ is fixed, φ and c are

arbitrary). Clearly B_ρ is either a point, a line, or a simply connected domain.

THEOREM 2. *Let*

$$(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{\Gamma\left(\frac{n}{\rho} + 1\right)}$$

be an entire function of order of growth ρ and let $H(\varphi) = \limsup \ln |F(re^{i\varphi})|/r^\rho$. If $f(\zeta)$ is the E_ρ -transform of $F(z)$, B is the set of its singular points, and B' is the set consisting of B and the point $\zeta = 0$, then

$$\sup_{\substack{\zeta \in B_\rho, \\ |\arg \zeta + \varphi| \leq \frac{\pi}{\rho}}} \operatorname{Re} \zeta^\rho e^{i\rho\varphi} \leq H(\varphi) \leq \sup_{\substack{\zeta \in B'_\rho, \\ |\arg \zeta + \varphi| \leq \frac{\pi}{\rho}}} \operatorname{Re} \zeta^\rho e^{i\rho\varphi} \quad (7)$$

(B_ρ and B'_ρ are the sets corresponding to B and B').

Proof: The reasoning used to prove Theorem 1 also establishes the left side of (7). To prove the right side we write:

$$F(re^{i\varphi}) = \frac{1}{2\pi i} \int_C f(\zeta) E_\rho(re^{i\varphi}\zeta) d\zeta.$$

Since the contour C can be chosen arbitrarily close to the boundary of B_ρ , then by the same reasoning as in Theorem 1 we obtain:

$$H(\varphi) = \overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\varphi})|}{r^\rho} \leq \sup_{\zeta \in B_\rho} \overline{\lim}_{r \rightarrow \infty} \frac{\ln |E_\rho(r\zeta e^{i\varphi})|}{r^\rho}.$$

But we know the behavior of $E_\rho(z)$ for large z (Chapter I, Section 4, example 4)

$$\begin{aligned} E_\rho(z) &\sim \rho e^{z^\rho}, & |\arg z| &\leq \frac{\pi}{2\rho} - \eta, \\ E_\rho(z) &\sim \frac{a}{z}, & \frac{\pi}{2\rho} + \eta &\leq |\arg z| \leq \pi. \end{aligned}$$

Therefore,

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |E_p(r\zeta e^{i\varphi})|}{r\rho} = \operatorname{Re} \zeta^\rho e^{i\rho\varphi}, \quad |\arg \zeta e^{i\varphi}| \leq \frac{\pi}{2\rho},$$

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln |E_p(r\zeta e^{i\varphi})|}{r\rho} = 0, \quad \frac{\pi}{2\rho} \leq |\arg \zeta e^{i\varphi}| \leq \pi.$$

Since the point $\zeta = 0$ is in the set B_ρ , it follows that $\sup \operatorname{Re} \zeta^\rho e^{i\rho\varphi}$ ($\zeta \in B_\rho$) cannot be negative for any φ . Therefore

$$\sup_{\zeta \in B'_\rho} \operatorname{Re} \zeta^\rho e^{i\rho\varphi} = \sup_{\zeta \in B_\rho} \overline{\lim}_{r \rightarrow \infty} \frac{\ln |E_p(r\zeta e^{i\varphi})|}{r\rho},$$

and the theorem is proven.

Theorem 2 shows that the set of singularities of $f(\zeta)$ tells nothing about the behaviour of $F(z)$ in those directions in which $F(z)$ does not grow. For $\rho < \frac{1}{2}$ this has no meaning, but for $\rho < \frac{1}{2}$ this is a real deficiency in the theorem.

This is due to the fact that the indicator of $E_\rho(z)$ is not equal to $\cos \rho \varphi$ for all φ ($|\varphi| \leq \pi$), but is only equal to $\cos \rho \varphi$ for $|\varphi| \leq \pi/2\rho$ and is zero for all other φ . At first glance it might seem that this is accidental, due to a bad choice of $\Phi(z)$. However, this is not the case. It can be shown that even if $\Phi(z)$ has very regular growth, in most cases the indicator of $\Phi(z)$ will still be equal to $\cos \rho \varphi$ only for $|\varphi| \leq \pi/2\rho$, and will be zero for other φ . There are some exceptions, for example, $\mu(x) = e^{-x}$.

A major deficiency in the use of the Borel transform to study entire functions is that we have not succeeded in establishing a connection between the properties of $f(\zeta)$ and the distribution of zeros of $F(z)$. Results of this sort would be exceptionally valuable (even for the case of the classical Borel transform).

The Borel transform and the generalized transform can also be used to obtain asymptotic estimates for $F(z)$, if we can succeed in separating the principal part of $f(\zeta)$ in a neighborhood of those singular points that are farthest (in a given direction) from the point $\zeta = 0$.

CHAPTER III

Special Cases of Asymptotic Estimates

1. Estimates of entire functions of the form

$$F(z) = \int_0^{\infty} \mu(t) e^{tz} dt, \quad F(z) = \int_{-\infty}^{\infty} \mu(t) e^{tz} dt$$

In the preceding chapter we discussed in detail the conclusions that could be drawn from a knowledge of the precise asymptotic behavior of this or that element of an entire function. Our conclusions can be formulated as follows.

Knowing the coefficients, we can estimate the maximum modulus very precisely (if the coefficients all have the same sign).

Knowing the behavior of the function close to a direction in which the modulus is maximum, we can estimate the coefficients.

Knowing the zeros of the function and the nonvanishing factor, both the coefficients and the growth of the function can be estimated everywhere.

Thus a rather complete investigation of the behavior of the function at infinity is possible (using only the general theory), knowing only the zeros of the function.

However in most applications the zeros of the entire function are not given. Therefore the general theory of entire functions usually does not tell us how to obtain precise asymptotic estimates for concrete entire functions. To obtain such estimates special methods are usually employed. The basic tool in obtaining these estimates is the saddlepoint method, which can be applied as soon as

we have a suitable integral representation of the function. Thus this section could be regarded as a series of examples of the use of the saddlepoint method to estimate entire functions.

We shall try to choose examples that will illustrate methods of estimation in some of the most important cases. In each of these cases we have preferred examples rather than general theorems, since if we wished to preserve some generality in our theorems both the proofs and the statements would become quite complicated.

We begin by considering a class of functions encountered very frequently in analysis

$$F(z) = \int_0^{\infty} \mu(t) e^{tz} dt,$$

where $\mu(t)$ is an analytic function, regular on the positive real axis, and decreasing along it fast enough so that the integral $F(z)$ converges for all z .

Let us first make a crude qualitative investigation of the possibilities of the saddlepoint method.

First of all, $\ln \mu(t)$ tends to $-\infty$ faster than At . If $\mu(t)$ is regular in the right half-plane then by the Phragmen-Lindelöf principle we can assert that $\ln \mu(t)$ tends to $-\infty$ with the same rapidity in a certain tongue-shaped domain narrower than a half-plane. The faster $\ln \mu(t)$ tends to $-\infty$, the narrower is the domain. The function $|\mu(t)e^{tz}|$ for $\operatorname{Re} z > 0$ behaves as follows inside this domain: it is bounded at the origin ($t=0$), then increases (the increase is greatest toward the boundaries of the tongue-shaped domain), and then decreases. As $\operatorname{Re} z$ increases, the barrier of large values is higher and farther out. This is where the saddlepoint will be found, through which the contour of integration from zero to infinity must pass.

For z sufficiently close to the imaginary axis the height of the barrier may be comparable to the value of the integrand at the origin of the contour (that is, to 1). Then (as well as for z in the left half-plane) the asymptotic estimate will contain terms from the beginning of the contour. These considerations are usually sufficient if $\mu(t)$ decreases sufficiently rapidly. For example:

$$1. F(z) = \int_0^{\infty} u^{\alpha} e^{-e^u} e^{uz} du.$$

We shall apply the saddlepoint method, taking for $\varphi(u) = u^{\alpha}$ $h(z, u) = uz - e^u$. The tongue-shaped domain of greatest decrease of the integrand, containing the positive u -axis, will be the half-strip $\operatorname{Re} u > 0$, $|\operatorname{Im} u| < \frac{1}{2}\pi$. The saddlepoints are given by the equation $z = e^u$, that is, $u = \ln z + 2\pi ni$. Only one of them lies inside our tongue-shaped domain, namely the point:

$$u_0(z) = \ln z, \quad |\operatorname{Im} \ln z| < \frac{\pi}{2}.$$

The value of the integrand for $u = u_0(z)$ is

$$(\ln z)^{\alpha} e^{z \ln z - z},$$

which is essentially larger than unity inside any angle $|\arg z| \leq \frac{1}{2}\pi - \eta$.

Thus, using, for example, formula (3), Section 5, Chapter I, we obtain:

$$F(z) \sim (\ln z)^{\alpha} \sqrt{\frac{2\pi}{z}} e^{z \ln z - z} \sim (\ln z)^{\alpha} \Gamma(z),$$

$$|\arg z| \leq \frac{\pi}{2} - \eta, \quad \eta > 0. \quad (1)$$

For $\frac{1}{2}\pi + \eta \leq |\arg z| \leq \pi$ the asymptotics of $F(z)$ are

determined entirely by the beginning of the contour, so that we have:

$$F(z) \sim \frac{\Gamma(\alpha+1)}{z^{\alpha+1}}, \quad \frac{\pi}{2} + \eta \leq |\arg z| \leq \pi, \quad \eta > 0. \quad (2)$$

In the remaining angles the principal term in the asymptotics of $F(z)$ is given by the sum of the right sides of (1) and (2).

In case the growth of $-1n\mu(t)$ is somewhat different from t , then our reasoning must be supplemented. To see the reason for this, we consider another example:

$$2. F(z) = \int_0^{\infty} u^{\alpha-\rho} e^{uz} du.$$

In this case the tongue-shaped domain is slightly narrower than a half-plane, but wider than any angle of opening less than $\frac{1}{2}\pi$. The saddlepoints are given by the equation

$$\rho(\ln u + 1) = z,$$

that is, $u_0(z) = e^{(z-\rho)/\rho}$.

The value of the integrand at the saddlepoint is

$$\exp \left\{ \rho e^{\frac{z}{\rho}-1} + \frac{\alpha}{\rho} (z - \rho) \right\}.$$

It is clear that this value is much larger than unity in the half-strips

$$\operatorname{Re} z > 0, \quad -\frac{\pi}{2} + 2\pi n < \operatorname{Im} z < \frac{\pi}{2} + 2\pi n,$$

while in the other cases it is less than unity or comparable to it.

If one were to apply the preceding reasoning uncritically, then one might try to write the same estimates for

$F(z)$ in all the half-strips $\operatorname{Re} z > 0$, $|\operatorname{Im} z - 2\pi n| > \frac{1}{2}\pi$. However, this would be false.

The fact is that our reasoning assumed that the boundaries of the tongue-shaped domain were sufficiently high that two such domains of decrease could not be joined at infinity with decreasing e^{t^2} .

For $\mu(t) = u^{-\rho u}$ this is not the case. Indeed, consider the function $u^{-\rho u} e^{uz}$ on the circle $|u| = R$. Putting $u = Re^{i\varphi}$, $z = x + iy$ we have:

$$\begin{aligned} |u^{-\rho u} e^{uz}| &= \exp \operatorname{Re} \{-\rho Re^{i\varphi} (\ln R + i\varphi) + Re^{i\varphi} (x + iy)\} = \\ &= \exp \{R(x - \rho \ln R) \cos \varphi + R(\rho \varphi - y) \sin \varphi\}. \end{aligned}$$

From this formula it is easy to see that for $y > \pi \rho / 2$ the function $u^{-\rho u} e^{uz}$ tends to zero uniformly on the circle arc $|u| = R$, $0 \leq \arg u < \frac{1}{2}\pi$, and for $y < -\frac{1}{2}\pi \rho$ on the arc $|u| = R$, $-\frac{1}{2}\pi < \arg u < 0$. This means that in place of the real axis, we can integrate along the imaginary axis (that is, one of the boundaries of the tongue-shaped domain has been destroyed). There the integrand is bounded, and the asymptotics of $F(z)$ are determined entirely by the beginning of the contour.

This reasoning shows where it is necessary to take these estimates. Finally, we have in the half-strip $\operatorname{Re} z > 0$, $|\operatorname{Im} z| < \pi \rho / 2 - \eta$,

$$F(z) \sim \sqrt{2\pi\rho} e^{\left(\alpha + \frac{1}{2}\right) \frac{\pi - \rho}{\rho}} \exp \rho e^{\frac{1}{\rho} (z - \rho)} \quad (3)$$

For z lying outside the half-strip $\operatorname{Re} z > 0$, $|\operatorname{Im} z| < \pi \rho / 2 + \eta$,

$$F(z) \sim \frac{\Gamma(\alpha + 1)}{z^{\alpha + 1}}, \quad (4)$$

and finally, in the remaining parts of the plane the

asymptotics of $F(z)$ are given by the sum of these two estimates.

A similar destruction of the boundary of the tongue-shaped region can be observed whenever $\mu(t) = e^{-l(t)}$, where $l(t)$ is slowly decreasing.

These examples exhaust the relatively regular cases of asymptotic estimates of integrals of the form $\int \mu(t)e^{tz}dt$. In more irregular cases (for example, when the function $\mu(t)$ has singular points inside the tongue-shaped domain) one can sometimes combine the saddlepoint method with the theory of residues.

The general principle for estimating integrals of the form

$$F(z) = \int_0^{\infty} \mu(t) e^{tz} dt$$

may be crudely formulated as follows:

$F(z)$ can be estimated from the saddlepoint lying in the tongue-shaped domain provided that the value of the integrand at this point is much larger than unity. When the value of the integrand at this point becomes much smaller than unity, then at all points further from the real axis, the asymptotics of $F(z)$ are given by the beginning of the contour. At the remaining points of the plane the asymptotics are given by the sum of these two estimates.

The case

$$\int_{-\infty}^{\infty} \mu(t) e^{tz} dt.$$

is very similar to what we have studied. Here too one must find tongue-shaped domains, one containing the positive part, another the negative part of the real axis. Again if the growth of $\ln \mu(t)$ is substantially faster than that of t , as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$, then the asymptotics

of $F(z)$ are given by the principal saddlepoint (that is, the one at which the integrand takes the largest value), of the two saddlepoints, one in the right tongue-shaped domain and the other in the left domain (or by their sum).

The destruction of the boundary of the tongue-shaped domain occurs just as before, when $\ln \mu(t)$ grows only slightly faster than t .

We consider two examples:

$$3. F(z) = \int_{-\infty}^{\infty} e^{-t^{2q}} e^{tz} dt \quad (q \text{ is an odd integer}).$$

The tongue-shaped domains of fastest decrease of $\mu(t)$ are the sectors $|\arg t| < \pi/4q$, $|\arg t - \pi| < \pi/4q$. The saddlepoints are given by the equation $2qt^{2q-1} = z$, that is $t = (z/2q)^{1/(2q-1)}$. For $|\arg z| < \frac{1}{2}\pi$ only the point

$$t_1(z) = \left(\frac{z}{2q}\right)^{\frac{1}{2q-1}},$$

$$\arg t_1(z) = \frac{1}{2q-1} \arg z,$$

among all the saddlepoints, lies close to one of the tongue-shaped domains, namely the right one. For $|\arg z - \pi| < \frac{1}{2}\pi$ we have one saddlepoint close to the left domain, namely the point

$$t_2(z) = -\left(-\frac{z}{2q}\right)^{\frac{1}{2q-1}},$$

$$\arg(-t_2(z)) = \frac{1}{2q-1} \arg(-z).$$

Thus we obtain the estimate

$$F(z) \sim \sqrt{4\pi q(2q-1)} z^{-\frac{q-1}{2q-1}} e^{c_q z^{\frac{2q}{2q-1}}},$$

$$|\arg z| \leq \frac{\pi}{2} - \eta. \quad (5)$$

Here

$$c_q = \frac{2q-1}{2q} (2q)^{-\frac{1}{2q-1}}.$$

For $|\arg z - \pi| \leq \frac{1}{2}\pi - \eta$ we obtain the same formula (5) with z replaced by $-z$, since $F(z) = F(-z)$, while for $\frac{1}{2}\pi - \eta \leq |\arg z| \leq \frac{1}{2}\pi + \eta$ we obtain the asymptotic behavior of $F(z)$ by combining these two estimates.

Let us consider an example where the boundary of the tongue-shaped domain is destroyed:

$$4. \quad F(z) = \int_{-\infty}^{\infty} e^{-V\sqrt{1+t^2} \ln(1+t^2) + tz} dt$$

(the square root and logarithm are chosen to be positive for positive real t).

The tongue-shaped domains are the right and left half-planes. The equation for the saddlepoints is

$$z = \frac{t \ln(1+t^2)}{\sqrt{1+t^2}} + \frac{2t}{\sqrt{1+t^2}},$$

or

$$z = 2(1 + \ln t) \left(1 + O\left(\frac{1}{t^2}\right)\right), \quad \operatorname{Re} t > 0,$$

$$z = -2(1 + \ln(-t)) \left(1 + O\left(\frac{1}{t^2}\right)\right), \quad \operatorname{Re} t < 0.$$

This gives us an asymptotic expression for the saddlepoints

$$t_1(z) = e^{\left(\frac{z}{2}-1\right)(1+O(e^{-z}))} = e^{\frac{z}{2}-1} + O\left(ze^{-\frac{z}{2}}\right),$$

$$t_2(z) = -e^{\left(-\frac{z}{2}+1\right)(1+O(e^z))} = -e^{-\frac{z}{2}+1} + O\left(ze^{\frac{z}{2}}\right).$$

The point $t_1(z)$ lies in the right domain, the point $t_2(z)$ in the left. The point $t_1(z)$ gives us the estimates for $\operatorname{Re} z > 0$ and $t_2(z)$ for $\operatorname{Re} z < 0$. Since we have an even function it is sufficient to estimate $F(z)$ in the right half-plane.

The value of the integrand at the saddlepoint $t=t_1(z)$ is

$$e^{2\sigma} \frac{s}{2} - 1 + O\left(z e^{-\frac{s}{2}}\right) = e^{2\sigma} \frac{s}{2} - 1 \left(1 + O\left(z e^{-\frac{s}{2}}\right)\right).$$

This expression is essentially larger than unity in the strips

$$\operatorname{Re} z > 0, \quad -\pi + 4\pi n < \operatorname{Im} z < \pi + 4\pi n.$$

This means that for $\operatorname{Re} z > 0$, $|\operatorname{Im} z| < \pi - \eta$ we have for $F(z)$ the estimate

$$F(z) \sim \sqrt{\pi e} e^{-\frac{s}{4}} e^{2\sigma} \frac{s}{2} - 1. \quad (6)$$

To study what happens for $|\operatorname{Im} z| > \pi$ we consider the modulus of the integrand. Putting $t = r e^{i\varphi}$, $z = x + iy$, we have for $0 \leq \varphi \leq \frac{1}{2}\pi$:

$$\begin{aligned} \ln |u(t) e^{tz}| &= -2(r \ln r \cos \varphi - r \varphi \sin \varphi) + \\ &+ r x \cos \varphi - r y \sin \varphi + O\left(\frac{\ln r}{r}\right) \\ &= r \cos \varphi (x - 2 \ln r) + r \sin \varphi (2\varphi - y) + O\left(\frac{\ln r}{r}\right). \quad (7) \end{aligned}$$

From this estimate it is clear that, for $y > \pi$ the integrand tends to zero like e^{-ar} , uniformly in the quadrant $0 \leq \varphi \leq \frac{1}{2}\pi$.

By symmetry the same will hold in the quadrant $\frac{1}{2}\pi \leq \varphi \leq \pi$. This means that for $y > \pi$ we can replace the contour of integration by an arbitrary contour lying in the upper half-plane (provided, of course, that it does not contain the singular point $t=i$). In particular, if we take for the contour the segment of the imaginary axis from i to ∞ , described twice, we obtain:

$$F(z) = i \int_1^{\infty} e^{-i V \sqrt{u^2-1} \ln(u^2-1) + \pi V \sqrt{u^2-1}} e^{iuz} du - \\ - i \int_1^{\infty} e^{i V \sqrt{u^2-1} \ln(u^2-1) + \pi V \sqrt{u^2-1}} e^{iuz} du.$$

For $\text{Im } z > \pi$ these integrals converge and the largest value of the integrand is attained at the end of the contour (that is, for $u=1$). Therefore the asymptotic estimate for $F(z)$ is given by the difference of the estimates for each of the two integrals. The two integrals can each be estimated separately by formula (2), Section 5, Chapter I, with some complications that arise from the singularity of the integrand at $u=1$. We have:

$$i \int_1^{\infty} e^{-i V \sqrt{u^2-1} \ln(u^2-1)} e^{\pi(V \sqrt{u^2-1}-u)} e^{(iz+\pi)u} du \sim \\ \sim \frac{e^{iz}}{z-\pi i} + \frac{e^{iz} \ln z}{(iz+\pi)^{3/2}},$$

$$i \int_1^{\infty} e^{i V \sqrt{u^2-1} \ln(u^2-1)} e^{\pi(V \sqrt{u^2-1}-u)} e^{(iz+\pi)u} du \sim \\ \sim \frac{e^{iz}}{(z-\pi i)} - \frac{e^{iz} \ln z}{(iz+\pi)^{3/2}},$$

from which we find that for $\text{Im } z \geq \pi + \eta$, $\eta > 0$, we have the estimate

$$F(z) \sim \frac{2e^{iz} \ln z}{(iz+\pi)^{3/2}} \sim 2e^{i\left(z+\frac{\pi}{4}\right)} z^{-\frac{3}{2}} \ln z. \quad (8)$$

For $\text{Im } z < -\pi - \eta$ we obtain a similar estimate using the evenness of $F(z)$, while for $\text{Re } z > 0$, $\pi - \eta \leq \text{Im } z \leq \pi + \eta$, $F(z)$ is estimated, as usual, by the sum of (6) and (8).

The general method of estimating integrals of the form

$$F(z) = \int_{-\infty}^{\infty} \mu(t) e^{tz} dt$$

can be formulated as follows:

The estimate for $F(z)$ is given by one of two saddle-points lying near the right or left tongue-shaped domains of uniform decrease of $\mu(t)$. However if $-\ln \mu(t)$ grows only slightly faster than t , then the integrand goes from large values through small values to large values again and the estimate for $F(z)$ is not obtained from the saddle-point but from changing the contour (the estimate results from "destroying the boundary").

By means of integrals of the form considered in this section it is easy to express integrals of the form

$$\int_0^{\infty} \mu(t) \cos tz dt, \quad \int_0^{\infty} \mu(t) \sin tz dt.$$

If $\mu(t)$ is regular on the entire real axis and is even, then $\int_0^{\infty} \mu(t) \cos tz dt$ can be expressed in terms of $\int_{-\infty}^{\infty} \mu(t) e^{tz} dt$, while in the opposite case it can be expressed in terms of $\int_0^{\infty} \mu(t) e^{tz} dt$.

2. The Poisson summation formula and estimates of functions of the form $F(z) = \sum_{n=0}^{\infty} \mu(n) z^n$

For functions with an integral representation it is comparatively easy to obtain asymptotic estimates by the saddlepoint method. The situation is more complicated if the function is not given by an integral, but by a finite or infinite series. The Euler-Maclaurin formula was used for similar estimates, but it can only be applied when the terms of the series do not vary too rapidly as

functions of n . We shall return to the Euler-Maclaurin formula a little later, but first we shall present another method for obtaining estimates of sums and series, somewhat simpler and cruder than the Euler-Maclaurin formula, but applicable to cases where the terms vary rapidly as functions of n .

THEOREM 1. *Let*

$$\int_0^{\infty} |F(x)| dx < \infty, \quad \int_0^{\infty} |F'(x)| dx < \infty.$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} F(n) &= \frac{1}{2} F(0) + \int_0^{\infty} F(x) dx + \\ &+ 2 \sum_{n=1}^{\infty} \int_0^{\infty} F(x) \cos 2\pi n x dx. \quad (1) \end{aligned}$$

If we require integrability of $|F(x)|$ and $|F'(x)|$ over the whole axis, then we have the formula

$$\sum_{n=-\infty}^{\infty} F(n) = \int_{-\infty}^{\infty} F(x) dx + 2 \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} F(x) \cos 2\pi n x dx.$$

Proof: First of all we can write

$$\sum_{n=0}^{\infty} F(n) = F(0) + \int_0^{\infty} F(x) d[x] = F(0) - \int_0^{\infty} F'(x) [x] dx.$$

Next we represent the function $[x]$ by the series

$$[x] = x - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin 2\pi n x,$$

which converges for all x , and whose partial sums remain bounded. The absolute integrability of $F'(x)$ permits us to multiply this series by $F'(x)$ and integrate the result term-by-term; we obtain:

$$\begin{aligned} \sum_{n=0}^{\infty} F(n) &= F(0) - \int_0^{\infty} x F'(x) dx + \\ &+ \frac{1}{2} \int_0^{\infty} F'(x) dx - \sum_{n=1}^{\infty} \frac{1}{\pi n} \int_0^{\infty} F'(x) \sin 2\pi n x dx. \end{aligned}$$

Integrating by parts we obtain (1).

Let us consider some simple examples of the application of this *Poisson formula*.

1. Find the asymptotic behavior in a neighborhood of the point $z=1$ of the function

$$f(z) = \sum_{n=1}^{\infty} n^{\alpha} z^n, \quad \operatorname{Re} \alpha > 0.$$

Let $F(x) = x^{\alpha} e^{-x}$; then

$$f(e^{-t}) = \sum_{n=0}^{\infty} F(n).$$

Since the conditions on $F(x)$ are satisfied, we have:

$$\begin{aligned} f(e^{-t}) &= \int_0^{\infty} x^{\alpha} e^{-xt} dx + 2 \sum_{n=1}^{\infty} \int_0^{\infty} x^{\alpha} e^{-xt} \cos 2\pi n x dx = \\ &= \sum_{n=-\infty}^{\infty} \int_0^{\infty} x^{\alpha} e^{-x(t+2\pi n i)} dx = \Gamma(\alpha+1) \sum_{n=-\infty}^{\infty} (t+2\pi n i)^{-\alpha-1} \end{aligned}$$

Passing to z we obtain:

$$f(z) = \Gamma(\alpha + 1) \sum_{n=-\infty}^{\infty} (-\ln z + 2\pi ni)^{-\alpha-1},$$

$$|\operatorname{Im} \ln z| \leq \pi. \quad (2)$$

Formula (2) shows that the only singularity of $f(z)$ in the entire z -plane is at $z=1$ ($z=0$ is not a singularity). The behavior of $f(z)$ near $z=1$ is given by the equation

$$f(z) = \frac{\Gamma(\alpha + 1)}{(-\ln z)^{\alpha+1}} + f_1(z),$$

where $f_1(z)$ is regular at $z=1$. Since

$$-\ln z = (1-z)f_0(z), \quad f_0(1) = 1,$$

where $f_0(z)$ is regular at $z=1$ we have:

$$f(z) = \frac{\Gamma(\alpha + 1)}{(1-z)^{\alpha+1}} f_0(z) + f_1(z), \quad f_0(1) = 1, \quad (3)$$

where $f_0(z)$ and $f_1(z)$ are regular at $z=1$. In formula (3) expand $f_0(z)$ and $f_1(z)$ in powers of $z-1$. In this way we can obtain as many terms in the asymptotic expansion of $f(z)$ about $z=1$ as we wish. (Note that this series will converge.)

$$2. \quad f(z) = \sum_{n=1}^{\infty} n^{\alpha} e^{i n \beta} z^n, \quad \operatorname{Re} \alpha > 0, \quad 0 < \beta < 1.$$

Proceeding as in Theorem 1 we find that

$$f(e^{-t}) = \sum_{n=-\infty}^{\infty} \int_0^{\infty} x^{\alpha} e^{i x \beta} e^{-(t+2\pi ni)x} dx.$$

Consider the function

$$\varphi(u) = \int_0^{\infty} x^{\alpha} e^{i x \beta} e^{-xu} dx.$$

For $u > 0$

$$\varphi(u) = \frac{1}{u^{\alpha+1}} \int_0^\infty y^\alpha e^{-y+i\frac{y^\beta}{u^\beta}} dy = \frac{1}{\beta u^{\alpha+1}} \int_0^\infty t^{\frac{\alpha-\beta+1}{\beta}} e^{-t} e^{it u^{-\beta}} dt,$$

that is

$$\varphi(u) = \frac{1}{\beta u^{\alpha+1}} \psi\left(\frac{t}{u^\beta}\right),$$

where $\Psi(\zeta)$ is the entire function:

$$\psi(\zeta) = \int_0^\infty t^\gamma e^{-t^q} e^{t\zeta} dt$$

which is of the form considered in the preceding section. If we apply the method developed there we easily obtain for $\Psi(\zeta)$ the asymptotic estimate

$$\begin{aligned} \psi(\zeta) &\sim \sqrt{2\pi} (q-1)^{-\frac{1}{2}} q^{\frac{1}{2}-\gamma} \zeta^{\frac{2\gamma-q+2}{2q-2}} \exp\left\{(q-1)q^{-\frac{q}{q-1}} \zeta^{\frac{q}{q-1}}\right\}, 4) \\ |\arg \zeta| &\leq \frac{\pi(q-1)}{2q} - \eta. \\ \psi(\zeta) &\sim \frac{\Gamma(\gamma+1)}{\zeta^{\gamma+1}}, \quad \frac{\pi(q-1)}{2q} + \eta \leq |\arg \zeta| \leq \pi \quad (5) \end{aligned}$$

(and, as usual, the sum of these quantities for the remaining ζ).

Consequently, about the point $z=1$ $f(z)$ has the form

$$f(z) = \frac{1}{\beta (-\ln z)^{\alpha+1}} \cdot \psi\left(\frac{t}{(-\ln z)^\beta}\right) + f_1(z),$$

where $\Psi(\zeta)$ is an entire function of ζ , and $f_1(z)$ is regular at $z=1$. Using (4) and (5) it is easy to find the asymptotic behavior of $f(z)$ near the point $z=1$. Thus, for example, for z close to one in any angle $|\arg(1-z)| \leq \frac{1}{2}\pi - \eta$ we obtain for $f(z)$ the estimate

$$f(z) \sim \frac{1}{\beta} \Gamma\left(\frac{\alpha+1}{\beta}\right) e^{-\frac{\pi(\alpha+1)}{\beta} i} + f_1(1), \quad (6)$$

since the angle $|\arg(1-z)| \leq \frac{1}{2}\pi - \eta$ corresponds to a ζ -domain in which (5) is valid.

The Poisson formula can be successfully applied to obtain asymptotic estimates of entire functions of the form

$$f(z) = \sum_{n=0}^{\infty} \mu(n) z^n.$$

The principle is very simple. By the Poisson formula

$$f(e^{\zeta}) = \frac{F(0)}{2} + \sum_{n=-\infty}^{\infty} F(\zeta + 2\pi n i), \quad F(\zeta) = \int_0^{\infty} \mu(t) e^{\zeta t} dt \quad (7)$$

(the convergence of the series is taken in the sense of principal value). Since e^{ζ} has period $2\pi i$, it is sufficient to investigate the behavior of $f(e^{\zeta})$ for $\operatorname{Re} \zeta > 0$, $|\operatorname{Im} \zeta| < \pi$. We can find an asymptotic estimate for $F(z)$ by the method of Section 1. In most important cases the principal term will be just the term $F(\zeta)$, and all other terms will be much smaller.

We consider first the simplest case of such an estimate:

$$3. f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\ln(n+2))^n}.$$

For the function $F(\zeta)$ (in the notations of formula (7)) it is easy to obtain an estimate by the method of Section 1. We have

$$F(\zeta) \sim \sqrt{2\pi t_0(\zeta) \ln t_0(\zeta)} e^{\frac{t_0(\zeta)}{\ln t_0(\zeta)}},$$

$$\zeta = \ln \ln(t_0(\zeta) + 2) + O\left(\frac{1}{\ln t_0(\zeta)}\right),$$

while $\operatorname{Re} t_0(\zeta)/\ln t_0(\zeta) > 0$, $|\operatorname{Im} t_0(\zeta)| < \frac{1}{2}\pi$ for the given ζ and for all ζ closer to the positive real axis.

If $|\operatorname{Im} t_0(\zeta)| > \frac{1}{2}\pi$ (and for all ζ further away), then

$$F(\zeta) = \frac{1}{\zeta} + O\left(\frac{1}{\zeta^2}\right).$$

Thus, under the condition $\operatorname{Re} t_0(\zeta) > 0$, $|\operatorname{Im} t_0(\zeta)| < \frac{1}{2}\pi$ (for the given ζ and for all ζ closer to the positive real axis) we obtain:

$$f(e^\zeta) \sim \sqrt{2\pi t_0(\zeta) \ln t_0(\zeta)} e^{\frac{t_0(\zeta)}{\ln t_0(\zeta)}} + \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{2\pi n i + \zeta} + O\left(\frac{1}{\zeta}\right),$$

since if $|\operatorname{Im} \zeta| < \pi$ then the second estimate certainly works for $F(\zeta + 2\pi n i)$. Simplifying, and using the fact that $t_0(\zeta) = \exp(e^\zeta) - 2 + O(e^{-\zeta})$, we find:

$$f(e^\zeta) \sim \sqrt{2\pi e^\zeta} e^{\frac{1}{2}e^\zeta} e^{e^\zeta - \zeta} + O(1),$$

from which finally

$$f(z) \sim \sqrt{2\pi z} e^{\frac{z}{2}} e^{e^z - \ln z}, \quad \operatorname{Re} z > 0, \quad |\operatorname{Im} z| \leq \frac{\pi}{2}. \quad (8)$$

If $F(\zeta) = 1/\zeta + O(1/\zeta^2)$, then

$$f(e^\zeta) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{\zeta + 2\pi n i} + O\left(\frac{1}{\zeta}\right) = O(1),$$

that is, $f(z) = O(1)$ outside the half-strip $\operatorname{Re} z > 0$, $|\operatorname{Im} z| > \frac{1}{2}\pi$. Together with (8) this provides the complete solution to our problem.

Naturally the question arises to what extent we can improve the estimates obtained by this method. The fact is that there are many possibilities of improving the estimates of $F(\zeta)$ —the saddlepoint method enables us to

obtain an asymptotic series for $F(\zeta)$, both where the asymptotics of $F(\zeta)$ are determined by the saddlepoint and where they are determined by the beginning of the contour. As regards the asymptotics of $f(z)$, at first glance they might seem to coincide with the asymptotics of $F(\ln z)$ where the latter are large, and where $F(\zeta) \sim \sum a_n/\zeta^{n+1}$ we obtain $f(z) \sim \sum a_n/(\ln z)^n$. However this is not correct.

Indeed, when $F(\zeta) = 1/\zeta + O(1/\zeta^2)$ we obtain:

$$f(\zeta) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{1}{\zeta + 2\pi n i} + O\left(\frac{1}{\zeta}\right),$$

but

$$\sum_{n=-\infty}^{\infty} \frac{1}{\zeta + 2\pi n i} = \frac{1}{\zeta} + \sum_{n=1}^{\infty} \frac{2\zeta}{\zeta^2 + 4\pi^2 n^2} = \frac{1}{2} \frac{1 + e^{-\zeta}}{1 - e^{-\zeta}} = \frac{1}{2} + O(e^{-\zeta})$$

for $\operatorname{Re} \zeta > 0$. In exactly the same manner we can show that all the other terms are much smaller than they appear at first glance. This means that where

$$F(\zeta) \sim \sum_{n=0}^{\infty} \frac{a_n}{\zeta^{n+1}}, \quad f(z) = O((\ln z)^{-k})$$

for all k . This estimate is better than the one obtained in example 3, but it does not even give the first term in the asymptotic expansion. To obtain the asymptotics of $f(z)$ where it is not increasing it is better to use a modification of the Euler-Maclaurin formula.

THEOREM 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} \mu(n) z^n,$$

where $\mu(t)$ is regular in the half-plane $\operatorname{Re} t \geq 0$ and satisfies the conditions:

1. along the positive real axis

$$\lim_{t \rightarrow \infty} \frac{\ln |\mu(t)|}{t} = -\infty;$$

2. $|\mu(t)e^{t(x+iy)}| < M(x)e^{-\epsilon|t|}$ for $y \geq \eta$ in the angle $0 \leq \arg(t-\sigma) < \frac{1}{2}\pi$; $|\mu(t)e^{t(x+iy)}| < M(x)e^{-\epsilon|t|}$ for $y < -\eta$ in the angle $-\frac{1}{2}\pi \leq \arg(t-\sigma) \leq 0$ (here $\eta < \pi$, $\sigma < 0$, $-N-1 < \sigma < -N$, where N is an integer).

Then for $|\operatorname{Im} \zeta| \leq 2\pi - \eta$

$$f(e^\zeta) = \int_{\sigma}^{\infty} \mu(t) e^{t\zeta} dt - \sum_{k=1}^N \mu(-k) e^{-k\zeta} + \\ + \int_{\sigma}^{\sigma+i\infty} \frac{\mu(t) e^{t\zeta}}{e^{-2\pi it} - 1} dt - \int_{\sigma-i\infty}^{\sigma} \frac{\mu(t) e^{t\zeta}}{e^{2\pi it} - 1} dt. \quad (9)$$

Sometimes the following formula is convenient

$$f(e^\zeta) = - \sum_{k=1}^N \mu(-k) e^{-k\zeta} - \int_{\sigma-i\infty}^{\sigma+i\infty} \mu(t) e^{t\zeta} \frac{dt}{e^{-2\pi it} - 1}, \quad (10) \\ \eta \leq \operatorname{Im} \zeta \leq 2\pi - \eta.$$

Proof: Just as in the derivation of the Euler-Maclaurin formula let us consider the integral

$$\frac{1}{2i} \int_{C_R} \mu(t) e^{t\zeta} \operatorname{ctg} \pi t dt,$$

where C_R is the rectangle with the vertices $\sigma - i\rho$, $R + \frac{1}{2} - i\rho$, $R + \frac{1}{2} + i\rho$, $\sigma + i\rho$ (R an integer). If $\operatorname{Im} \zeta \geq \eta$, then

from 2) the limit of the integral exists as $R \rightarrow \infty$. This limit is equal to

$$\frac{1}{2i} \int_{\sigma-i\infty}^{\infty-i\infty} \mu(t) e^{t\zeta} \operatorname{ctg} \pi t dt - \frac{1}{2i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu(t) e^{t\zeta} \operatorname{ctg} \pi t dt.$$

On the other hand, the integral over C_R is equal to the sum of the residues. Finding this sum and its limit as $R \rightarrow \infty$ we obtain:

$$\begin{aligned} \sum_{n=-N}^{\infty} \mu(n) e^{n\zeta} &= \frac{1}{2i} \int_{\sigma-i\infty}^{\infty-i\infty} \mu(t) e^{t\zeta} \operatorname{ctg} \pi t dt - \\ &\quad - \frac{1}{2i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu(t) e^{t\zeta} \operatorname{ctg} \pi t dt. \end{aligned}$$

Applying the same considerations to the integral

$$\int_{C_R} \mu(t) e^{t\zeta} dt = 0,$$

we find that

$$\int_{\sigma-i\infty}^{\infty-i\infty} \mu(t) e^{t\zeta} dt - \int_{\sigma-i\infty}^{\sigma+i\infty} \mu(t) e^{t\zeta} dt = 0.$$

Therefore we have:

$$\begin{aligned} \sum_{n=-N}^{\infty} \mu(n) e^{n\zeta} &= \frac{1}{2i} \int_{\sigma-i\infty}^{\infty-i\infty} \mu(t) e^{t\zeta} (\operatorname{ctg} \pi t - i) dt - \\ &\quad - \frac{1}{2i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu(t) e^{t\zeta} (\operatorname{ctg} \pi t + i) dt + \int_{\sigma-i\infty}^{\infty-i\infty} \mu(t) e^{t\zeta} dt. \quad (11) \end{aligned}$$

If $|\operatorname{Im} \zeta| \leq 2\pi - \eta$, then the integrals in (11) converge and the first of them can be taken over any contour lying in the angle $-\frac{1}{2}\pi \leq \arg(t - \sigma) \leq 0$, by condition 2).

In particular we can take it along the contour $(\sigma - i\rho, \sigma - i\infty)$. Then we obtain:

$$\sum_{n=-N}^{\infty} \mu(n) e^{n\zeta} = \int_{\sigma-i\rho}^{\sigma-i\infty} \mu(t) e^{t\zeta} dt - \\ - \frac{1}{2l} \int_{\sigma-i\rho}^{\sigma+i\infty} \mu(t) e^{t\zeta} (\operatorname{ctg} \pi t + i) dt + \frac{1}{2l} \int_{\sigma-i\rho}^{\sigma-i\infty} \mu(t) e^{t\zeta} (\operatorname{ctg} \pi t - i) dt.$$

Noting that

$$\operatorname{ctg} \pi t + i = \frac{2l}{1 - e^{-2\pi i t}}, \quad \operatorname{ctg} \pi t - i = \frac{2l}{e^{2\pi i t} - 1},$$

and putting $\rho = 0$, we obtain:

$$\sum_{n=-N}^{\infty} \mu(n) e^{n\zeta} = \int_{\sigma}^{\infty} \mu(t) e^{t\zeta} dt + \int_{\sigma}^{\sigma+i\infty} \mu(t) e^{t\zeta} \frac{dt}{e^{-2\pi i t} - 1} - \\ - \int_{\sigma-i\infty}^{\sigma} \mu(t) e^{t\zeta} \frac{dt}{e^{2\pi i t} - 1},$$

that is, formula (9). Formula (10) is obtained in exactly the same way, except that in (11) one must subtract i from $\operatorname{ctg} \pi t$ in both integrals.

Let us consider some examples of the application of Theorem 2.

$$4. f(z) = \sum_{n=0}^{\infty} n^{-\frac{n}{\rho}} z^n, \quad \rho > \frac{1}{2}.$$

It is easy to verify that for $\mu(t) = t^{-1/\rho}$ conditions (1) and (2) are satisfied for all σ and for $\eta > \pi/2\rho$. However $\mu(t)$ is regular only for $\operatorname{Re} t > 0$, and therefore we must take $\sigma = 0$. Nonetheless in the last two integrals we can replace the rays

$$(0, 0 + i\infty), \quad (0 - i\infty, 0),$$

by the polygonal line contours

$$(0, -\gamma, -\gamma + i\infty), \quad (0, -\gamma, -\gamma - i\infty).$$

Thus we have succeeded, along with other advantages, in eliminating the complications arising from the divergence of the integral at $t=0$.

On the upper side of the slit $(0, -\gamma)$ we have

$$\frac{\mu(t) e^{t\zeta}}{e^{-2\pi i t} - 1} = e^{\frac{\sigma}{\rho} \ln x} e^{-x\zeta} \frac{e^{-\frac{\pi i x}{\rho}}}{e^{2\pi i x} - 1},$$

and on the lower side

$$\frac{\mu(t) e^{t\zeta}}{e^{2\pi i t} - 1} = e^{\frac{\sigma}{\rho} \ln x} e^{-x\zeta} \frac{e^{\frac{\pi i x}{\rho}}}{e^{-2\pi i x} - 1}.$$

Therefore from (9) we obtain:

$$\begin{aligned} f(e^\zeta) &= \int_0^\infty e^{-\frac{t}{\rho} \ln t + t\zeta} dt \\ &\quad - \int_0^\gamma e^{\frac{\sigma}{\rho} \ln x - x\zeta} \left\{ \frac{e^{-\frac{\pi i x}{\rho}}}{e^{-2\pi i x} - 1} + \frac{e^{\frac{\pi i x}{\rho}}}{e^{2\pi i x} - 1} \right\} dx + \\ &\quad + \int_{-\gamma}^{-\gamma + i\infty} \mu(t) e^{t\zeta} \frac{dt}{e^{-2\pi i t} - 1} - \int_{-\gamma - i\infty}^{-\gamma} \mu(t) e^{t\zeta} \frac{dt}{e^{2\pi i t} - 1}. \end{aligned}$$

For the third and fourth integrals we have the obvious estimate $O(e^{-\pi t})$. Also, the second integral can easily be transformed into the form

$$\int_0^1 e^{\frac{1}{\rho} x \ln x} \frac{\sin \frac{\pi x}{\rho}}{\sin \pi x} e^{-x\zeta} dx \sim \frac{\rho+1}{\rho\zeta}.$$

Therefore,

$$f(e^\zeta) - \int_0^\infty e^{\frac{1}{\rho} t \ln t + t\zeta} dt \sim -\frac{\rho+1}{\rho\zeta},$$

$$|\operatorname{Im} \zeta| \leq 2\pi - \frac{\pi}{2\rho} - \eta, \quad \eta > 0.$$

Combining this estimate with the estimate obtained for $F(\zeta)$ in example 2, Section 1, we obtain:

$$f(z) \sim \sqrt{\frac{2\pi z}{\rho e}} \exp \frac{\rho z^\rho}{e}, \quad |\arg z| \leq \frac{\pi}{2\rho} - \eta,$$

$$f(z) \sim -\frac{1}{\rho \ln z}, \quad \frac{\pi}{2\rho} + \eta \leq |\arg z| \leq \pi$$

and the sum of these estimates at other points.

$$5. f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\rho}, \quad \rho > \frac{1}{2}.$$

With the aid of Stirling's formula it is easy to see that the function $\mu(t) = [\Gamma(t+1)]^{-1/\rho}$ satisfies conditions (1) and (2) for all σ and $\eta > \pi/2\rho$. However, since $\mu(t)$ has a singular point at $t = -1$, in formula (9) we can take $\sigma = -1 + \epsilon$, $\epsilon > 0$.

By formula (9)

$$\begin{aligned}
 f(e^z) &= \int_{\sigma}^{\infty} [\Gamma(t+1)]^{-\frac{1}{\rho}} e^{tz} dt = \\
 &= \int_{\sigma}^{\sigma+i\infty} [\Gamma(t+2)]^{\frac{1}{\rho}} (t+1)^{\frac{1}{\rho}} e^{tz} \frac{dt}{e^{-2\pi it} - 1} - \\
 &\quad - \int_{\sigma-i\infty}^{\sigma} [\Gamma(t+2)]^{-\frac{1}{\rho}} (t+1)^{\frac{1}{\rho}} e^{tz} \frac{dt}{e^{2\pi it} - 1}.
 \end{aligned}$$

Each of the last two integrals converges for $\sigma = -1$. Therefore, putting $\sigma = -1$ and finding the asymptotics as $\zeta \rightarrow \infty$ we obtain:

$$\begin{aligned}
 f(e^z) &= \int_{-1}^{\infty} [\Gamma(t+2)]^{-\frac{1}{\rho}} e^{tz} dt \sim -\frac{1}{2\pi i} \int_{-1}^{-\infty + 0. i} (t+1)^{\frac{1}{\rho}-1} e^{tz} dt + \\
 &\quad + \frac{1}{2\pi i} \int_{-1}^{-\infty - 0. i} (t+1)^{\frac{1}{\rho}-1} e^{tz} dt \sim -\frac{\sin \frac{\pi}{\rho}}{\pi} e^{-z} z^{-\frac{1}{\rho}}.
 \end{aligned}$$

Thus for the final estimate of $f(z)$ we must find the asymptotic behavior of

$$F(\zeta) = \int_{-1}^{\infty} [\Gamma(t+1)]^{-\frac{1}{\rho}} e^{t\zeta} dt,$$

which is easily done by the method of Section 1.

We do not write down this estimate for the places where $f(z)$ is increasing, but merely note the estimate for $\pi/2\rho + \eta \leq |\arg z| \leq \pi$:

$$f(z) \sim \left(1 - \frac{1}{\pi} \sin \frac{\pi}{\rho}\right) \frac{1}{z (\ln z)^{\frac{1}{\rho}}}.$$

From what has been said it is clear that the estimate for $f(z)$ where it is not increasing (assuming that $\mu(t)$ satisfies conditions (1) and (2)), becomes more accurate for larger values of $-\sigma$ for which $\mu(t)$ is regular in the half-plane $\operatorname{Re} t > \sigma$. The estimate takes its simplest form when $\mu(t)$ is an entire function. Then

$$f(z) \sim - \sum_{n=1}^{\infty} \frac{\mu(-n)}{z^n}.$$

For example, if

$$\mu(t) = -\frac{1}{\Gamma\left(\frac{t}{\rho} + 1\right)},$$

that is, $f(z) = E_{\rho}(z)$, the function that we encountered in example 4, Section 4, Chapter I, then

$$f(z) \sim - \sum_{n=1}^{\infty} \frac{1}{z^n \Gamma\left(-\frac{n}{\rho} + 1\right)}, \quad \frac{\pi}{2\rho} + \eta \leq |\arg z| \leq \pi,$$

from which, using the relationship $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$, we obtain:

$$f(z) \sim -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n}{\rho}\right) \sin \frac{\pi n}{\rho}}{z^n}, \quad \frac{\pi}{2\rho} + \eta \leq |\arg z| \leq \pi,$$

which is in complete agreement with the estimate obtained for $E_{\rho}(z)$ in Section 4, Chapter I.

Using the results of this section we can partially solve the question of the indicator of the function $\Phi(z)$ defining the generalized Borel transform

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{m(n)}, \quad m(z) = \int_0^{\infty} x^z \mu(x) dx.$$

Indeed, if $\mu(t)$ has a regular structure then we can easily find the indicator of $\Phi(z)$ using the methods of Section 1 and of this section, and we can convince ourselves that it is equal to $\cos \rho \varphi$ for $|\varphi| \leq \pi/2\rho$ and to zero for the other values of φ , unless $1/m(\zeta)$ is an entire function with zeros at $-1, -2, -3, \dots$. The solution of the indicator question in this case is somewhat more complicated.

3. Functions of the form $F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{p(n)}\right)$,

$$F(z) = \sum_{n=1}^{\infty} \frac{q(n)}{z + p(n)}$$

In Section 4, Chapter II we considered the question of the behavior at infinity of canonical products with regularly distributed zeros. In Chapter II we were not interested in delicate estimates, but we wished to obtain them as simply as possible. Therefore we did not stop to obtain the further terms in the asymptotic series, or to investigate the behavior of $F(z)$ close to directions in which the zeros were distributed, etc. In this section we shall obtain more delicate asymptotic properties of canonical products and of certain other functions.

For our first problem we shall attempt to find all the terms in the asymptotic expansion of $\ln F(z)$,

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{p(n)}\right),$$

valid in any angle $|\arg z'| \leq \pi - \eta$, $\eta > 0$.

We make the following assumptions about the function $\mu(t)$:

- (1) $\mu(t) > 0$, $\mu'(t) > 0$, $0 < t < \infty$;
 (2) $\mu(t) = t^{1/\rho} l(t)$, where $l(t)$ is regular in the plane with the slit $(-\infty, 0)$, and $l'(t)/l(t) \rightarrow 0$ as $t \rightarrow \infty$;

$$(3) \int_0^{\infty} \frac{dt}{\mu(t)} < \infty.$$

To obtain the asymptotic series we shall use a method that we have already encountered twice: in the derivation of the Euler-Maclaurin formula and in the proof of Theorem 2, Section 2 of this chapter.

Choose a contour $C_{N,\theta}$, symmetric about the real axis, consisting of a segment of the line $\operatorname{Re} t = N + \frac{1}{2}$ and two curves $C_{N,\theta}^+$ and $C_{N,\theta}^-$. The curve $C_{N,\theta}^+$ is a part of a curve C_θ^+ , going from the point $t = \theta$ to infinity in the upper half-plane, and meeting each line $\operatorname{Re} t = A$, $A > \theta$, exactly once. The curve $C_{N,\theta}^-$ is symmetric to $C_{N,\theta}^+$ in the real axis. Consider the integral

$$I_{N,\eta} = \frac{1}{2i} \int_{C_{N,\eta}} \ln \left[1 + \frac{z}{\mu(t)} \right] \operatorname{ctg} \pi t \, dt.$$

If there are no points t for which $\mu(t) = -z$ inside the contour $C_{N,\theta}$, then

$$I_{N,\eta} = \sum_{k=1}^N \ln \left[1 + \frac{z}{\mu(n)} \right].$$

If $\mu(t)$ satisfies conditions (1) and (2), then we can always choose the curves $C_\theta^\pm$ so that if z is any point lying inside the angle $|\arg z| \leq \pi - \eta$, $\eta > 0$, then there are no points t inside the contour $C_{N,\theta}$ for which $\mu(t) = -z$. In addition this can be done in such a manner that some angle $|\arg t| < \beta$ can be placed between the curves C_θ^+ and C_θ^- (this follows from the fact that $\mu(re^{i\varphi}) \sim o^{i\varphi/\rho} \mu(r)$, by condition (2)).

Passing to the limit as $N \rightarrow \infty$ (this is possible by condition (3)), we obtain:

$$\begin{aligned} \ln F(z) &= \frac{1}{2l} \int_{C_{\theta}^{-}} \ln \left[1 + \frac{z}{\mu(t)} \right] \operatorname{ctg} \pi t \, dt - \\ &\quad - \frac{1}{2l} \int_{C_{\theta}^{+}} \ln \left[1 + \frac{z}{\mu(t)} \right] \operatorname{ctg} \pi t \, dt = I_{\theta}^{-} - I_{\theta}^{+} \quad (1) \end{aligned}$$

(the integration along C_{θ}^{+} and C_{θ}^{-} is taken in both cases from $t = \theta$ to $t = \infty$). The further transformations are carried out as in Section 2. We have:

$$\begin{aligned} I_{\theta}^{+} &= \frac{1}{2l} \int_{C_{\theta}^{+}} \ln \left[1 + \frac{z}{\mu(t)} \right] (\operatorname{ctg} \pi t + i) \, dt - \frac{1}{2} \\ &\quad \int_{C_{\theta}^{+}} \ln \left[1 + \frac{z}{\mu(t)} \right] dt = \\ &= - \int_{C_{\theta}^{+}} \ln \left[1 + \frac{z}{\mu(t)} \right] \frac{dt}{e^{-2\pi i t} - 1} - \frac{1}{2} \int_{\theta}^{\infty} \ln \left[1 + \frac{z}{\mu(t)} \right] dt, \quad (2) \end{aligned}$$

since

$$\int_{C_{\theta}^{+}} \ln \left[1 + \frac{z}{\mu(t)} \right] dt = \int_{C_{\theta}^{-}} \ln \left[1 + \frac{z}{\mu(t)} \right] dt = \int_{\theta}^{\infty} \ln \left[1 + \frac{z}{\mu(t)} \right] dt$$

by the regularity of $\ln [1 + z/\mu(t)]$ between C_{θ}^{+} and C_{θ}^{-} . Similarly

$$I_{\theta}^{-} = \int_{C_{\theta}^{-}} \ln \left[1 + \frac{z}{\mu(t)} \right] dt + \frac{1}{2} \int_{\theta}^{\infty} \ln \left[1 + \frac{z}{\mu(t)} \right] dt. \quad (3)$$

Substituting (2) and (3) into (1) we find:

$$\ln F(z) = \int_0^{\infty} \ln \left[1 + \frac{z}{\mu(t)} \right] dt + \int_{c_0^+} \ln \left[1 + \frac{z}{\mu(t)} \right] \frac{dt}{e^{-2\pi i t} - 1} + \\ + \int_{c_0^-} \ln \left[1 + \frac{z}{\mu(t)} \right] \frac{dt}{e^{2\pi i t} - 1}. \quad (4)$$

It is easy to see that the first of the integrals on the right side of (4) is the principal term as $z \rightarrow \infty$. The sum of the last two integrals can be expanded in an asymptotic series in powers of $1/z$. Indeed, writing

$$\ln \left[1 + \frac{z}{\mu(t)} \right] = \ln z - \ln \mu(t) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{[\mu(t)]^k}{z^k}$$

and integrating term-by-term, we obtain:

$$\ln F(z) = \int_0^{\infty} \ln \left[1 + \frac{z}{\mu(t)} \right] dt + a(\theta) + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{c_k(\theta)}{z^k},$$

where

$$c_0(\theta) = \int_{c_0^+} \frac{\ln \mu(t) dt}{e^{-2\pi i t} - 1} + \int_{c_0^-} \frac{\ln \mu(t) dt}{e^{2\pi i t} - 1}, \\ kc_k(\theta) = \int_{c_0^+} \frac{[\mu(t)]^k dt}{e^{-2\pi i t} - 1} + \int_{c_0^-} \frac{[\mu(t)]^k dt}{e^{2\pi i t} - 1}, \\ a(\theta) = \int_{c_0^+} \frac{dt}{e^{-2\pi i t} - 1} + \int_{c_0^-} \frac{dt}{e^{2\pi i t} - 1}.$$

The quantity $a(\theta)$ can be easily calculated: $a(\theta) = \theta - \frac{1}{2}$.

The principal term can be put into a more convenient form by integrating by parts:

$$\begin{aligned} \int_0^\infty \ln \left[1 + \frac{z}{\mu(t)} \right] dt &= -\theta \ln \left[1 + \frac{z}{\mu(\theta)} \right] + z \int_0^\infty \frac{t\mu'(t) dt}{\mu(t)[z + \mu(t)]} = \\ &= z \int_{\mu(\theta)}^\infty \frac{\nu(u) du}{u(u+z)} - \theta \ln \left[1 + \frac{z}{\mu(\theta)} \right]. \end{aligned}$$

Here $\nu(u)$ is the inverse function to $\mu(t)$. Thus we obtain the final result:

$$\begin{aligned} \ln F(z) \sim z \int_{\mu(\theta)}^\infty \frac{\nu(u) du}{u(u+z)} + \theta \ln \frac{z\mu(\theta)}{z + \mu(\theta)} - \frac{1}{2} \ln z + \\ + \sum_{k=0}^\infty (-1)^{k+1} \frac{c_k(\theta)}{z^k}. \end{aligned}$$

$$c_0(\theta) = \int_{C_\theta^+} \frac{\ln \mu(t)}{e^{-2\pi i t} - 1} dt + \int_{C_\theta^-} \frac{\ln \mu(t)}{e^{2\pi i t} - 1} dt,$$

$$c_k(\theta) = \frac{1}{k} \int_{C_\theta^+} \frac{[\mu(t)]^k}{e^{-2\pi i t} - 1} dt + \frac{1}{k} \int_{C_\theta^-} \frac{[\mu(t)]^k}{e^{2\pi i t} - 1} dt.$$

In the formulae for $c_k(\theta)$ it is better to take the integration along the ray $(\theta, \theta + i\infty)$ instead of along the contour C_θ^+ , and along the ray $(\theta, \theta - i\infty)$ instead of along C_θ^- .

Note finally that conditions (1) and (2), and also the form of the function, were not essential in applying the method.

Let us consider some examples.

$$1. \quad \Gamma_\rho(z) = \prod_{n=1}^\infty \left(1 + zn^{-\frac{1}{\rho}} \right), \quad 0 < \rho < 1.$$

Formula (5) can be applied directly to this case, so that here our only problem is to calculate the coefficients in that formula. The best value of Θ to take is $\Theta = 0$. For the principal term we have:

$$z \int_0^{\infty} \frac{v(u) du}{u(u+z)} = z \int_0^{\infty} \frac{u^{\rho-1} du}{u+z} = \frac{\pi z^{\rho}}{\sin \pi \rho}.$$

For $c_k(0)$ formula (5) gives us:

$$\begin{aligned} c_k(0) &= \frac{1}{k} \int_0^{i\infty} \frac{t^{\frac{k}{\rho}} dt}{e^{-2\pi i t} - 1} + \frac{1}{k} \int_0^{-i\infty} \frac{t^{\frac{k}{\rho}} dt}{e^{2\pi i t} - 1} = \\ &= -\frac{2}{k} \sin \frac{k\pi}{2\rho} \int_0^{\infty} \frac{y^{\frac{k}{\rho}}}{e^{2\pi y} - 1} dy = \\ &= -\frac{2}{k} \sin \frac{k\pi}{2\rho} \int_0^{\infty} \sum_{m=1}^{\infty} e^{-2\pi m y} y^{\frac{k}{\rho}} dy = \\ &= -\frac{2}{k} \Gamma\left(\frac{k}{\rho} + 1\right) \zeta\left(\frac{k}{\rho} + 1\right) (2\pi)^{-\frac{k}{\rho}-1} \sin \frac{k\pi}{2\rho}, \end{aligned}$$

or, by the functional equation for $\zeta(s)$,

$$\zeta(s) = 2(2\pi)^{s-1} \Gamma(1-s) \zeta(1-s) \sin(\pi s/2),$$

we have

$$c_k(0) = \frac{1}{k} \zeta\left(-\frac{k}{\rho}\right).$$

(We could have used the formula that we obtained in Section 2, Chapter I in example 1.)

For $c_0(0)$ recall that the formula

$$A(f, \theta) = \int_{\theta}^{\theta+i\infty} \frac{f(z) dz}{e^{-2\pi i z} - 1} + \int_{\theta}^{\theta-i\infty} \frac{f(z) dz}{e^{2\pi i z} - 1}$$

served to define the constant in the Euler-Maclaurin formula, that is, $A(f, \Theta)$ is the limit of the difference be-

tween the sums $\Sigma^* f(k)$ and the increasing terms of the series as $n \rightarrow \infty$.

Therefore

$$\begin{aligned} & \int_0^{\theta+i\infty} \frac{\ln t^{\frac{1}{\rho}}}{e^{-2\pi i t} - 1} dt + \int_{\frac{\theta}{2}}^{\theta-i\infty} \frac{\ln t^{\frac{1}{\rho}}}{e^{2\pi i t} - 1} dt = \\ &= \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \ln k^{\frac{1}{\rho}} - \int_{\frac{\theta}{2}}^n \ln t^{\frac{1}{\rho}} dt - \frac{1}{2} \ln t^{\frac{1}{\rho}} \right\} = \\ &= \frac{1}{\rho} \lim_{n \rightarrow \infty} \ln \frac{n! e^n}{n^{n+\frac{1}{2}}} \left(\frac{\theta}{e} \right)^{\frac{1}{\rho}} = \frac{1}{2\rho} \ln 2\pi - \frac{1}{\rho} (\theta \ln \theta - \theta). \end{aligned}$$

That is,

$$c_0(\theta) = \frac{1}{2\rho} \ln 2\pi.$$

Thus, for $|\arg z| \leq \pi - \eta$, $\eta > 0$,

$$\ln \Gamma_{\frac{1}{\rho}}(z) \sim \frac{\pi z^{\frac{1}{\rho}}}{\sin \pi \frac{1}{\rho}} - \frac{1}{2} \ln z - \frac{1}{2\rho} \ln 2\pi - \sum_{k=1}^{\infty} (-1)^k \frac{\zeta\left(-\frac{k}{\rho}\right)}{k z^k} \quad (6)$$

$$2. F(z) = \prod_{n=1}^{\infty} (1 + z e^{-\sqrt[n]{n}}).$$

Formula (5) cannot be applied directly to this function, as conditions (1) and (2) on $\mu(t)$ are not satisfied. However the reasoning can be carried over with no essential change. Indeed, for C_{θ}^+ one can take a path consisting of a part of the line $\operatorname{Re} t = \theta$ and an arc of the parabola $\operatorname{Im} \sqrt{t} = \eta$. Then the part of the plane between C_{θ}^+ and C_{θ}^- is mapped by the function $e^{\sqrt{t}}$ inside the angle $|\arg e^{\sqrt{t}}| \leq \eta$. Therefore for $|\arg z| \leq \pi - \eta_1$, $\eta_1 > \eta$, there are no

points t between C_0^+ and C_0^- for which $\exp(1/z) = -z$. Consequently all further transformations are legitimate, and by formula (5) we obtain the asymptotic series for $\ln F(z)$. It remains to choose the best value of Θ and to compute the integrals that are obtained. We begin with the principal term. (We take Θ equal to zero.) We have:

$$\begin{aligned}\varphi(z) &= z \int_{\mu(\frac{1}{z})}^{\infty} \frac{\nu(u) du}{u(u+z)} = z \int_1^{\infty} \frac{\ln^2 u du}{u(u+z)} = \\ &= \int_{\frac{1}{z}}^{\infty} \frac{\ln^2 uz du}{u(u+1)} = \int_0^{\frac{1}{z}} \frac{(\ln z - \ln t)^2}{t+1} dt.\end{aligned}$$

For $\varphi(z)$ it is easy to obtain the equation

$$[z\varphi'(z)]' = \frac{2}{z} \ln(z+1) = \frac{2 \ln z}{z} + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kz^{k+1}}.$$

Integrating, we find:

$$\begin{aligned}\varphi(z) &= \frac{1}{3} \ln^3 z + a_1 \ln z + a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 z^k}, \\ a_1 &= \varphi'(1) + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} - 2 \int_0^1 \frac{\ln t}{t+1} dt, \\ a_0 &= \int_0^1 \frac{\ln^2 t}{t+1} dt - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}.\end{aligned}$$

Since

$$\int_0^1 \frac{\ln t}{t+1} dt = \int_0^\infty \frac{u du}{e^u + 1} = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^2},$$

$$\int_0^1 \frac{\ln^2 t}{t+1} dt = \int_0^\infty \frac{u^2 du}{e^u + 1} = 2 \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^3},$$

we have

$$a_0 = 0, \quad a_1 = 4 \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^2} = 4 \left(1 - \frac{1}{4}\right) \zeta(2) = \frac{\pi^2}{2},$$

and the final form of the principal term is

$$\varphi(z) = \frac{1}{3} \ln^3 z + \frac{\pi^2}{2} \ln z + 2 \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^3 z^k}. \quad (7)$$

For $c_0(0)$, as in the preceding example, we obtain:

$$c_0(0) = \zeta\left(-\frac{1}{2}\right).$$

Finally

$$c_k(0) = \lim_{\Theta \rightarrow 0} c_k(\Theta) = -\frac{1}{2} + 2 \int_0^\infty \frac{\sin k \sqrt{y}}{e^{2\pi y} - 1} dy$$

(in passing to the limit $\Theta \rightarrow 0$ it is necessary to subtract half the residue of the function $\exp(\sqrt{t}) \operatorname{ctg} \pi t$ at $t=0$ from the value of the integral for $\Theta=0$).

We obtain finally for $|\arg z| \leq \pi - \eta$, $\eta > 0$,

$$\ln F(z) \sim \frac{1}{3} \ln^3 z + \frac{\pi^2 - 1}{2} \ln z - \zeta\left(-\frac{1}{2}\right) + \sum_{k=1}^\infty \frac{(-1)^{k+1} c_k}{k z^k}, \quad (8)$$

where

$$c_k = -\frac{1}{2} + \frac{2}{k^2} + 2 \int_0^{\infty} \frac{\sin k \sqrt{y}}{e^{2\pi y} - 1} dy.$$

Using this method there is no difficulty in obtaining all the terms in the asymptotic expansion of $\ln F(z)$, where $F(z)$ is a canonical product of any finite order. Since the reasoning is the same, we give only the final result.

THEOREM 1. *Let the function $\mu(t)$ satisfy conditions (1) and (2), let $\nu(u)$ be the inverse function to $\mu(t)$, let $0 < \theta < 1$, and let p be the smallest integer such that*

$$\int \nu(u) u^{-p-2} du < \infty.$$

Then for the entire function

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\mu(n)}\right) \exp \left\{ \sum_{m=1}^p \frac{(-1)^m}{m} \left[\frac{z}{\mu(n)} \right]^m \right\}$$

we have the asymptotic formula

$$\begin{aligned} \ln F(z) \sim & (-1)^p z^{p+1} \int_{\mu(\theta)}^{\infty} \frac{\nu(u) du}{u^{p+1}(u+z)} + \\ & + \theta \left\{ \ln \frac{z\mu(\theta)}{z + \mu(\theta)} + \sum_{m=1}^p \frac{(-1)^m}{m} \left[\frac{z}{\mu(\theta)} \right]^m \right\} - \\ & - \frac{1}{2} \ln z + \sum_{k=-p}^{\infty} \frac{(-1)^{k+1} c_k}{z^k}, \quad (9) \end{aligned}$$

where

$$c_0(\theta) = \int_{\sigma_\theta^+} \frac{\ln \mu(t)}{e^{-2\pi i t} - 1} dt + \int_{\sigma_\theta^-} \frac{\ln \mu(t)}{e^{2\pi i t} - 1} dt,$$

$$c_k(\theta) = \frac{1}{k} \int_{\sigma_\theta^+} \frac{[\mu(t)]^k}{e^{-2\pi i t} - 1} dt + \frac{1}{k} \int_{\sigma_\theta^-} \frac{[\mu(t)]^k}{e^{2\pi i t} - 1} dt.$$

Let us consider an example of the application of this method to functions of a different form.

$$3. F(z) = \sum_{n=1}^{\infty} \frac{n^{\alpha-1} \ln n}{n+z}, \quad 0 < \alpha < 1.$$

If we carry out all the steps in the derivation of (5), we obtain, for $|\arg z| \leq \pi - \eta$, $\eta > 0$, the equation

$$F(z) = \int_0^{\infty} \frac{t^{\alpha-1} \ln t}{t+z} dt + \int_{\sigma_\theta^+} \frac{t^{\alpha-1} \ln t}{t+z} \frac{dt}{e^{-2\pi i t} - 1} + \int_{\sigma_\theta^-} \frac{t^{\alpha-1} \ln t}{t+z} \frac{dt}{e^{2\pi i t} - 1},$$

from which we find:

$$F(z) \sim \int_0^{\infty} \frac{t^{\alpha-1} \ln t}{t+z} + \sum_{k=0}^{\infty} \frac{(-1)^k c_k(\theta)}{z^{k+1}},$$

$$c_k(\theta) = \int_{\sigma_\theta^+} \frac{t^{\alpha+k-1} \ln t}{e^{-2\pi i t} - 1} dt + \int_{\sigma_\theta^-} \frac{t^{\alpha+k-1} \ln t}{e^{2\pi i t} - 1} dt. \quad (10)$$

Take $\theta=0$ and compute the integral for the principal term. We have:

$$\begin{aligned} \int_0^{\infty} \frac{t^{\alpha-1} \ln t}{t+z} dt &= z^{\alpha-1} \ln z \int_0^{\infty} \frac{t^{\alpha-1} dt}{t+1} + z^{\alpha-1} \int_0^{\infty} \frac{t^{\alpha-1} \ln t}{t+1} dt = \\ &= \frac{\pi z^{\alpha-1} \ln z}{\sin \pi \alpha} - \frac{\pi^2 \cos \pi \alpha}{\sin^2 \pi \alpha} z^{\alpha-1}. \end{aligned}$$

In computing $c_k(0)$ note that $c_k(0) = (\partial / \partial a) \Upsilon_k(a, \Theta)$, where

$$\gamma_k(\alpha, \theta) = \int_{C_\theta^+} \frac{t^{\alpha+k-1} dt}{e^{-2\pi i t} - 1} + \int_{C_\theta^-} \frac{t^{\alpha+k-1}}{e^{2\pi i t} - 1} dt = \zeta(1-\alpha-k) + \frac{\theta^{\alpha+k}}{\alpha+k}.$$

Thus, for $|\arg z| \leq \pi - \eta$, $\eta > 0$,

$$F(z) \sim \frac{\pi z^{\alpha-1} \ln z}{\sin \pi \alpha} - \frac{\pi^2 \cos \pi \alpha}{\sin^2 \pi \alpha} z^{\alpha-1} + \sum_{k=1}^{\infty} (-1)^k \frac{\zeta'(1-\alpha-k)}{z^k} \quad (11)$$

Up till now we have only obtained asymptotic expansions that were good away from the zeros of the entire function (or from the poles of the meromorphic function). However the method we have presented also enables us to obtain the asymptotic behavior near these exceptional points. To this end we shall study carefully functions of the form

$$F(z) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{\mu(n) - z},$$

where $\varphi(t)$ and $\mu(t)$ satisfy the conditions:

(4) The functions $\varphi(t)$ and $\mu(t)$ are regular in the half-plane $\operatorname{Re} t \geq \Theta$, $0 < \Theta < 1$, and the integral

$$\int_0^{\infty} \frac{\varphi(t)}{\mu(t)} dt$$

is absolutely convergent along any path in this half-plane.

(5) The function $\mu(t)$ is real and positive for $t > 0$ and is schlicht in some domain D containing some half-strip $\operatorname{Re} t > A$, $|\operatorname{Im} t| < a$.

By $\nu(u)$ we mean that branch of the inverse function to $\mu(t)$ that is positive for positive u .

As at the beginning of this section we write:

$$I_0 = \frac{1}{2i} \int_{C_0^-} \frac{\varphi(t)}{\mu(t) - z} \operatorname{ctg} \pi t \, dt - \\ - \frac{1}{2i} \int_{C_0^+} \frac{\varphi(t)}{\mu(t) - z} \operatorname{ctg} \pi t \, dt = I_0^- - I_0^+. \quad (12)$$

We make the additional assumption that C_0^+ and C_0^- both lie inside D . If z is such that between the curves C_0^+ and C_0^- there is a point t at which $\mu(t) = z$, that is, $t = \nu(z)$ (by the schlichtness of $\mu(t)$ between C_0^+ and C_0^- there is at most one such point), then by the theory of residues

$$I_0 = \sum_{n=1}^{\infty} \frac{\varphi(n)}{\mu(n) - z} + \pi \frac{\varphi(\nu(z))}{\mu'(\nu(z))} \operatorname{ctg} \pi \nu(z) = \\ = F(z) + \pi \nu'(z) \varphi(\nu(z)) \operatorname{ctg} \pi \nu(z). \quad (13)$$

Let z , and therefore also $\nu(z)$, lie in the upper half-plane. Then, carrying out the usual transformations, we have:

$$\left. \begin{aligned} I_0^+ &= - \int_{C_0^+} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{-2\pi i t} - 1} - \frac{1}{2} \int_0^{\infty} \frac{\varphi(t) \, dt}{\mu(t) - z} + \\ &\quad + \pi i \varphi(\nu(z)) \nu'(z), \\ I_0^- &= \int_{C_0^-} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{2\pi i t} - 1} + \frac{1}{2} \int_0^{\infty} \frac{\varphi(t) \, dt}{\mu(t) - z}. \end{aligned} \right\} \quad (14)$$

From (12), (13) and (14) we obtain:

$$F(z) = \int_{\eta}^{\infty} \frac{\varphi(t) dt}{\mu(t) - z} + \pi i \frac{\nu'(z) \varphi(\nu(z))}{1 - e^{-2\pi i \nu(z)}} + \\ + \int_{C_{\eta}^{+}} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{-2\pi i t} - 1} + \int_{C_{\eta}^{-}} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{2\pi i t} - 1}. \quad (15)$$

Similarly, for z lying in the lower half-plane we obtain:

$$F(z) = \int_{\eta}^{\infty} \frac{\varphi(t) dt}{\mu(t) - z} - \pi i \frac{\nu'(z) \varphi(\nu(z))}{1 - e^{2\pi i \nu(z)}} + \\ + \int_{C_{\eta}^{+}} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{-2\pi i t} - 1} + \int_{C_{\eta}^{-}} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{2\pi i t} - 1}, \quad (16)$$

while for real z

$$F(z) = \int_{\eta}^{\infty} \frac{\varphi(t) dt}{\mu(t) - z} + \pi \nu'(z) \varphi(\nu(z)) \operatorname{ctg} \pi \nu(z) + \\ + \int_{C_{\eta}^{+}} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{-2\pi i t} - 1} + \int_{C_{\eta}^{-}} \frac{\varphi(t)}{\mu(t) - z} \frac{dt}{e^{2\pi i t} - 1} \quad (17)$$

(the first integral is a principal value).

Note that the last two integrals in (15), (16) and (17) are the same as those obtained earlier, and can be expanded in exactly the same way in asymptotic series. Thus the asymptotic behavior of $F(z)$ near the poles differs from the behavior away from them only in the second term of formulae (15), (16) and (17). It is easy to see that this term only plays a role near the poles, and on the ray $\arg z = \eta$ it decreases exponentially.

Let us see what these formulae tell us about the function of example 3.

$$4. F(z) = \sum_{n=1}^{\infty} \frac{n^{\alpha-1} \ln n}{n-z}, \quad 0 < \alpha < 1.$$

We only consider the case $z > 0$. In example 3 we already found all the coefficients in the asymptotic series. The new term has the form $\pi x^{\alpha-1} \ln x \operatorname{ctg} \pi x$. The principal term must be counted again, because the integral is a principal value. But the principal value is equal to half the sum of the limiting values from the two sides, and so

$$\text{prin. val.} \int_0^{\infty} \frac{t^{\alpha-1} \ln t}{t-x} dt = -\pi \operatorname{ctg} \pi \alpha x^{\alpha-1} \ln x - \frac{\pi^2}{\sin^2 \pi \alpha} x^{\alpha-1}.$$

This means, finally, for $x < 0$

$$F(x) \sim \pi x^{\alpha-1} \ln x (\operatorname{ctg} \pi x - \operatorname{ctg} \pi \alpha) - \frac{\pi^2 x^{\alpha-1}}{\sin^2 \pi \alpha} + \\ + \sum_{k=1}^{\infty} \frac{\zeta'(1-\alpha-k)}{x^k}.$$

From (15), (16) and (17) it is easy to obtain asymptotic formulae for the behavior of a canonical product near its zeros. We present the result for the case of real z .

THEOREM 2. *Let the function $\mu(t)$ satisfy conditions (1) and (2) and be schlicht in some domain D containing some half-strip $\operatorname{Re} t > A$, $|\operatorname{Im} t| < \alpha$, and let $\nu(u)$ be the function inverse to $\mu(t)$. Then for the entire function*

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\mu(n)}\right) \exp \left\{ \sum_{m=1}^p \frac{(-1)^m}{m} \left[\frac{z}{\mu(n)} \right]^m \right\}$$

we have, for $x > 0$, the asymptotic formula

$$\begin{aligned} \ln F(-x) \sim \ln \sin \pi \nu(x) - x^{p+1} \int_{\mu(\theta)}^{\infty} \frac{\nu(u) du}{u^{p+1}(u-x)} + \\ + \theta \left\{ \ln \frac{x^{\mu(\theta)}}{x - \mu(\theta)} + \sum_{m=1}^p \frac{1}{m} \left[\frac{x}{\mu(\theta)} \right]^m \right\} - \\ - \frac{1}{2} \ln x + a(\theta) + \sum_{k=-p}^{\infty} \frac{c_k(\theta)}{x^k}. \quad (18) \end{aligned}$$

Here

$$\begin{aligned} a(\theta) = \ln F(\mu(\theta)) - \ln \sin \pi \theta - \\ - \int_0^{\infty} \left\{ \ln \left(1 - \frac{\mu(\theta)}{\mu(t)} \right) + \sum_{m=1}^p \frac{1}{m} \left[\frac{\mu(\theta)}{\mu(t)} \right]^m \right\} dt - \\ - \int_{c_{\theta}^{+}} \left\{ \ln \left(1 - \frac{\mu(\theta)}{\mu(t)} \right) + \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{\mu(\theta)}{\mu(t)} \right]^m \right\} \frac{dt}{e^{-2\pi i t} - 1} - \\ - \int_{c_{\theta}^{-}} \left\{ \ln \left(1 - \frac{\mu(\theta)}{\mu(t)} \right) + \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{\mu(\theta)}{\mu(t)} \right]^m \right\} \frac{dt}{e^{2\pi i t} - 1}, \end{aligned}$$

$c_k(\theta)$ are the same as in (9), and the integral is a principal value.

In exactly the same way formulae can be given for the behavior of $\ln F(z)$ above and below the axis.

4. Asymptotic estimates of the zeros of entire functions

In Sections 1 and 2 we encountered certain types of entire functions admitting very exact asymptotic estimates as $z \rightarrow \infty$. In most cases these estimates are sufficiently precise that we can also obtain asymptotic formulae for the zeros of the functions. By giving a number of examples we shall demonstrate a method for doing this.

We begin with the simplest example.

1. $F(z) = J_0(z)$, the Bessel function of zero order.

To obtain an asymptotic formula for λ_n —the n th zero in magnitude of $J_0(z)$ —we use the asymptotic formula for $J_0(x)$:

$$J_0(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) + O\left(\frac{e^{|\operatorname{Im} z|}}{z^{3/2}}\right), \quad (1)$$

obtained by us in Section 5, Chapter I (example 3).

It is clear that for large n λ_n is near to one of the zeros of $\cos(z - \pi/4)$, that is, $\lambda_n = 3\pi/4 + \pi m + o(1)$ (where m depends in some manner on n).

We must find the dependence of m on n , and also estimate the order of the remainder term.

We shall first solve the easier problem, namely the second one.

In (1) put $z = \lambda_n$. We obtain:

$$\cos\left(\lambda_n - \frac{\pi}{4}\right) = O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{m}\right),$$

or

$$\sin\left(\lambda_n - \frac{3\pi}{4} - \pi m\right) = O\left(\frac{1}{m}\right),$$

from which we have:

$$\lambda_n = \frac{3\pi}{4} + \pi m + O\left(\frac{1}{m}\right).$$

It remains to determine the relation between m and n . For this purpose we find the number of zeros of $J_0(z)$ in the rectangle C_m with the vertices

$$\begin{aligned} im + \pi(m+1), \quad im - \pi(m+1), \\ -im - \pi(m+1), \quad -im + \pi(m+1). \end{aligned}$$

Since $J_0(z)$ is an even function, the number of zeros of $J_0(z)$ in C_m is equal to $2n$, and we can write:

$$2n = \frac{1}{2\pi i} \int_{C_m} \frac{J'_0(z)}{J_0(z)} dz.$$

Since (1) can be differentiated, on the contour C_m we have:

$$\frac{J'_0(z)}{J_0(z)} = -\operatorname{ctg}\left(z - \frac{\pi}{4}\right) + O\left(\frac{e^{|\operatorname{Im} z|}}{z}\right).$$

The integral over C_m of the last term will be $o(1)$, which means that

$$2n = -\frac{1}{2\pi i} \int_{C_m} \operatorname{ctg}\left(z - \frac{\pi}{4}\right) dz + o(1) = 2m + o(1)$$

(the integral of $-\operatorname{ctg}(z - \pi/4)$ over C_m is equal to the number of zeros of $\cos(z - \pi/4)$ inside C_m , that is, $2m$). If two integers differ by less than unity, they must be identical. Therefore $m = n$ for sufficiently large n . Consequently,

$$\lambda_n = \pi n + \frac{3\pi}{4} + O\left(\frac{1}{n}\right). \quad (2)$$

If we have an asymptotic formula for $J_0(z)$ with a large number of terms, then we can also find more terms in the formula for λ_n .

Our second example is somewhat more complicated.

$$2. F(z) = E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\frac{n}{\rho} + 1\right)}, \quad \rho > 1.$$

In Section 4, Chapter I (example 4) we obtained for the function $E_\rho(z)$ the asymptotic formula

$$\left. \begin{aligned} E_\rho(z) &\sim \rho e^{z^\rho} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n}{\rho}\right) \sin \frac{\pi n}{\rho}}{z^n}, & |\arg z| \leq \frac{\pi}{\rho}, \\ E_\rho(z) &\sim -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n}{\rho}\right) \sin \frac{\pi n}{\rho}}{z^n}, & \frac{\pi}{\rho} \leq |\arg z| \leq \pi. \end{aligned} \right\} 3)$$

The second formula in (3) shows that all sufficiently distant zeros of $E_\rho(z)$ lie in the angle $|\arg z| < \pi/\rho$.

In the preceding example the leading term was the function $\cos(z - \pi/4)$, whose zeros are well known. Here the leading term is the function

$$F_0(z) = \rho e^{z^\rho} - \frac{1}{\pi z} \Gamma\left(\frac{1}{\rho}\right) \sin \frac{\pi}{\rho}, \quad |\arg z| \leq \frac{\pi}{2\rho} + \eta.$$

Therefore our first task is to obtain an asymptotic formula for the zeros of $F_0(z)$ in the angle $|\arg z| < \pi/\rho$. The zeros of $F_0(z)$, as well as those of $E_\rho(z)$, are symmetric about the real axis and are distributed near the rays $\arg z = \pm \pi/2\rho$. Consider the curve given by the equation

$$|ze^{z^\rho}| = c_\rho, \quad c_\rho = \frac{\Gamma\left(\frac{1}{\rho}\right) \sin \frac{\pi}{\rho}}{\pi \rho}. \quad (4)$$

Let L denote that part of the curve that begins at the point

$$z = a_0 e^{\frac{\pi i}{\rho}}, \quad a_0 e^{-a_0^\rho} = c_\rho, \quad a_0 > 0,$$

and goes to the right from it, approaching asymptotically to the ray $\arg z = \pi/2\rho$.

All the zeros of $F_0(z)$ that interest us lie on the curve L_ρ . We number them in the order of increasing distance along the curve L_ρ . (We shall denote the zeros of $F_0(z)$ by $z_n = r_n \exp(i\varphi_n)$.) In addition we shall single out the points $z'_n = r'_n \exp(i\varphi'_n)$ on the curve L_ρ , at which

$$z'_n e^{z_n^\rho} = -c_\rho.$$

Clearly, $F_0(z'_n) = -2\rho c / z'_n$. The points z_n and z'_n occur alternately.

On the curve L the function $zF_0(z)$ behaves like a cosine, it oscillates and is bounded by the constant $2\rho c_\rho$.

To obtain asymptotic formulae for z_n and z'_n we use the equation of the curve L . First of all we have from (4):

$$r^\rho \cos \rho\varphi = -\ln r + \ln c_\rho$$

for all points $z = re^{i\varphi}$ lying on L_ρ . Hence for large r we have:

$$\varphi = \frac{\pi}{2\rho} + \frac{\ln r}{\rho r^\rho} - \frac{\ln c_\rho}{r^\rho} + O\left(\frac{(\ln r)^3}{r^{3\rho}}\right). \quad (5)$$

Equation (5) enables us to determine φ in terms of r .

To express r_n and r'_n in terms of n we proceed as follows.

On the curve L_ρ consider the function

$$\ln ze^{z^\rho},$$

where we have chosen that branch of the logarithm that is positive for positive z . In passing from $z=0$ to $z=a_0 e^{\pi i/\rho}$

along the ray $\arg z = \pi/\rho$, the function $\ln \exp(z^\rho)$ remains real, while in passing from $z = a_0 e^{\pi i}$ to $re^{i\varphi}$ the value of the imaginary part of this logarithm increases by $2\pi n + \rho \arg ze^{-\pi i/\rho}$, where n denotes the number of points on the curve L_ρ between $a_0 e^{\pi i/\rho}$ and $re^{i\varphi}$ at which the function $\exp(z^\rho)$ is real and positive. As z moves along L_ρ , $\arg z \exp(z^\rho)$ increases. But at the point $z = a_0 e^{\pi i/\rho}$ this argument is equal to π/ρ , at the point z_1' it is equal to π , and at z_n it is 2π . This means, first of all, that $r_{n-1} < r_n' < r_n$ for sufficiently large n , and secondly,

$$\left. \begin{aligned} r_n^\rho \sin \rho \varphi_n &= -\varphi_n + 2\pi n, \\ r_n'^\rho \sin \rho \varphi_n' &= -\varphi_n' + (2n+1)\pi. \end{aligned} \right\} \quad (6)$$

By combining (5) and (6) we can find an asymptotic formula for z_n and z_n' .

Indeed, replacing $\sin \rho \varphi$ and φ by their expressions in r we obtain:

$$\begin{aligned} 2\pi n &= r_n^\rho + \frac{\pi}{2\rho} + O\left(\frac{(\ln r_n')^\rho}{r_n'^\rho}\right), \\ 2\pi n &= r_n'^\rho + \pi + \frac{\pi}{2\rho} + O\left(\frac{(\ln r_n)^\rho}{r_n^\rho}\right). \end{aligned}$$

From these formulae we obtain asymptotic formulae for r_n and r_n' :

$$\begin{aligned} r_n &= (2\pi n)^{\frac{1}{\rho}} - \frac{(2\pi)^{\frac{1}{\rho}}}{4\rho^2} n^{\frac{1}{\rho}-1} + O\left(n^{\frac{1}{\rho}-2} (\ln n)^2\right), \\ r_n' &= (2\pi n)^{\frac{1}{\rho}} - \frac{(2\pi)^{\frac{1}{\rho}} (2\rho+1)}{4\rho^2} n^{\frac{1}{\rho}-1} + O\left(n^{\frac{1}{\rho}-2} (\ln n)^2\right). \end{aligned} \quad (7)$$

Substituting these expressions for r_n and r'_n into (5) we obtain asymptotic formulae for φ_n and φ'_n :

$$\begin{aligned}\varphi_n = \varphi'_n + O\left(\frac{\ln n}{n^2}\right) &= \frac{\pi}{2\rho} + \frac{1}{2\pi\rho^2} \frac{\ln n}{n} - \\ &- \frac{1}{2\pi} \left(\ln c_\rho - \frac{1}{\rho^2} \ln 2\pi \right) + O\left(n^{\frac{1}{\rho}-2} (\ln n)^2\right).\end{aligned}$$

It is now easy to obtain the final asymptotic estimates for z_n and z'_n :

$$\begin{aligned}z_n = (2\pi)^{\frac{1}{\rho}} e^{\frac{\pi i}{2\rho}} n^{\frac{1}{\rho}} + i \frac{(2\pi)^{\frac{1}{\rho}-1}}{\rho^2} n^{\frac{1}{\rho}-1} \ln n - \\ - \frac{(2\pi)^{\frac{1}{\rho}}}{\rho^2} \left\{ \frac{1}{4} e^{\frac{\pi i}{2\rho}} + i \frac{\rho^2 \ln c_\rho - \ln 2\pi}{2\pi} \right\} n^{\frac{1}{\rho}-1} + \\ + O\left(n^{2\left(\frac{1}{\rho}-1\right)} (\ln n)^2\right). \quad (8)\end{aligned}$$

$$\begin{aligned}z'_n = (2\pi)^{\frac{1}{\rho}} e^{\frac{\pi i}{2\rho}} n^{\frac{1}{\rho}} + i \frac{(2\pi)^{\frac{1}{\rho}-1}}{\rho^2} n^{\frac{1}{\rho}-1} \ln n - \\ - \frac{(2\pi)^{\frac{1}{\rho}}}{\rho^2} \left\{ \frac{2\rho+1}{4} e^{\frac{\pi i}{2\rho}} + i \frac{\rho^2 \ln c_\rho - \ln 2\pi}{2\pi} \right\} n^{\frac{1}{\rho}-1} + \\ + O\left(n^{2\left(\frac{1}{\rho}-1\right)} (\ln n)^2\right). \quad (9)\end{aligned}$$

Now that we have found an asymptotic formula for z_n , we must find which zeros of $E_\rho(z)$ are asymptotically close to z_n . To do this we first find the number N of zeros of $E_\rho(z)$ whose modulus does not exceed $r'_n = |z'_n|$. We have:

$$N = \frac{1}{2\pi i} \int_{|z|=r'_n} \frac{E'_\rho(z)}{E_\rho(z)} dz.$$

On the circle $|z|=r'_n$ the function $E_\rho(z)$ behaves as follows: first, on the whole circle $|E_\rho(z)| > a/|z|$, second, for $|\arg z| \leq \varphi'_n$

$$E_\rho(z) = \rho e^{z^\rho} \left(1 + O\left(\frac{e^{-z^\rho}}{z}\right) \right),$$

and finally, for $\varphi'_n + \eta \leq |\arg z| \leq \pi$, $E_\rho(z) = -\rho c_\rho/z + O(1/z^2)$. Therefore,

$$\frac{E'_\rho(z)}{E_\rho(z)} = \begin{cases} \rho z^{\rho-1} + O\left(\frac{e^{-z^\rho}}{z}\right), & |\arg z| \leq \varphi'_n, \\ -\frac{1}{z} + O\left(\frac{1}{z^2}\right), & \varphi'_n + \eta \leq |\arg z| \leq \pi. \end{cases} \quad (10)$$

On the remaining parts the estimate is more complicated. We shall come to it in a moment, but first we shall see what we get from the integral taken over those parts of the circle where (10) is valid. We have:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\substack{|z|=r'_n, \\ \varphi'_n + \eta \leq |\arg z| \leq \pi}} \frac{E'_\rho(z)}{E_\rho(z)} dz &= -1 + \frac{\varphi'_n + \eta}{\pi} + \\ &+ o(1) = -1 + \frac{1}{2\rho} + \eta_1 + o(1). \end{aligned}$$

by (6). Also,

$$\begin{aligned} \frac{1}{2\pi i} \int_{r'_n e^{-i\varphi'_n}}^{r'_n e^{i\varphi'_n}} \frac{E'_\rho(z)}{E_\rho(z)} dz &= \frac{1}{2\pi i} \int_{r'_n e^{-i\varphi'_n}}^{r'_n e^{i\varphi'_n}} \rho z^\rho dz + o(1) = \\ &= \frac{r_n^{\rho}}{\pi} \sin \rho \varphi'_n + o(1) = 2n - 1 - \frac{1}{2\rho} + o(1) \end{aligned}$$

Thus we see that the integral over those parts of the circle where (10) is valid gives us:

$$2n - 2 + O(\eta), \quad \eta \rightarrow 0.$$

It remains to estimate those parts of the integral where (10) does not hold. For $\varphi'_n \leq \arg z \leq \varphi'_n + \eta$ we have:

$$E_p(z) = \rho e^{z^p} - \frac{\rho c_p}{z} + O\left(\frac{1}{z^2}\right),$$

$$E'_p(z) = \rho^2 z^{p-1} e^{z^p} + \frac{\rho c_p}{z^2} + O\left(\frac{1}{z^3}\right).$$

This means that for $\varphi'_n \leq \arg z \leq \varphi'_n + \eta$, $|z| = r'_n$ (under these conditions $|z \exp(z^p) - c_p| > A z \exp(z^p)$) we have

$$\frac{E'_p(z)}{E_p(z)} = \frac{\rho z^p e^{z^p}}{z e^{z^p} - c_p} + O\left(\frac{1}{z}\right) = \frac{\rho z^p e^{z^p} + e^{z^p}}{z e^{z^p} - c_p} + O\left(\frac{1}{z}\right).$$

But, putting for simplicity $a_1 = r'_n \exp(i\varphi'_n)$, $a_2 = r'_n \exp(i(\varphi'_n + \eta))$, we obtain:

$$\begin{aligned} \frac{1}{2\pi i} \int_{a_1}^{a_2} \frac{E'_p(z)}{E_p(z)} dz &= \frac{1}{2\pi i} \ln \left(c_p - z'_n e^{z'^p_n} \right) + o(1) + O(\eta) = \\ &= \frac{1}{2\pi i} \ln 2c_p + o(1) + O(\eta) \end{aligned}$$

(since $z'_n \exp(z'^p_n) = -c_p$, by the choice of the point z'_n).

But the integral represents a real number. Therefore when the integral over (a_1, a_2) is added to the corresponding integral over the symmetric arc the imaginary parts must cancel.

Therefore for sufficiently large n

$$N = 2n - 2.$$

This means that for large n $E_\rho(z)$ has exactly $n-1$ zeros close to the ray $\arg z = \pi/2$ ρ inside the circle $|z| < r'_n$, and exactly $n-1$ zeros close to the ray $\arg z = -\pi/2$ ρ . Numbering the zeros that are close to the ray $\arg z = \pi \rho/2$ in the order of increasing moduli, and denoting them by λ_n , we find that λ_n itself is asymptotically close to z_n . This means that $\lambda_n = z_n + o(z_{n+1} - z_n)$.

Substituting λ_n into the formula

$$E_\rho(z) = F_0(z) + O\left(\frac{1}{z^2}\right),$$

we obtain:

$$F_0(\lambda_n) = O(\lambda_n^{-2}),$$

or

$$(\lambda_n - z_n) F'_0(z_n) = O\left(\frac{1}{z_n^2}\right),$$

which enables us to write the final asymptotic formula for λ_n :

$$\begin{aligned} \lambda_n = & (2\pi)^{\frac{1}{\rho}} e^{\frac{\pi i}{2\rho}} n^{\frac{1}{\rho}} + i \frac{(2\pi)^{\frac{1}{\rho}-1}}{\rho^2} n^{\frac{1}{\rho}-1} \ln n - \\ & - \frac{(2\pi)^{\frac{1}{\rho}}}{\rho^2} \left\{ \frac{1}{4} e^{\frac{\pi i}{2\rho}} + i \frac{\rho^2 \ln c_\rho - \ln 2\pi}{2\pi} \right\} n^{\frac{1}{\rho}-1} + O\left(n^{2(\frac{1}{\rho}-1)} (\ln n)^2\right). \end{aligned}$$

Here

$$c_\rho = \frac{\Gamma\left(\frac{1}{\rho}\right) \sin \frac{\pi}{\rho}}{\pi \rho}.$$

For $\frac{1}{2} < \rho < 1$ the zeros are estimated in exactly the same way, except that the various terms have different orders than they did before; for example, for $\frac{1}{2} < \rho < 1$ the

order of the remainder term in (11) is only less than the very first of the leading terms. For $\rho \leq \frac{1}{2}$ the estimate is even simpler, it is like the estimate for the zeros in example 1.

The method of estimation that we demonstrated in example 2 is characteristic for the general case. The fact that the principal part of the function, where it is increasing, was found with such great accuracy has no importance for us now. An estimate of the form $F(z) = A(z)(1 + O(1/z))$ would be entirely sufficient, and in practice such estimates can always be obtained.

The various remarks that we made when considering various entire functions of very regular structure can be summarized in the following general remarks on the structure of an arbitrary entire function of regular growth.

If $F(z)$ is an entire function of regular growth at infinity, then a neighborhood of the point $z = \infty$ can be divided into a countable number of tongue-shaped domains G_k (the common boundary of G_k and G_{k+1} will be denoted by L_k) such that strictly inside G_k we have $F(z) = \exp(h_k(z) (1 + o(1)))$ here $h_k(z)$ is regular in G_k and on L_k and L_{k-1} , while close to the curve L_k

$$F(z) = e^{h_k(z)}(1 + o(1)) + e^{h_{k+1}(z)}(1 + o(1)).$$

The only condition on the function $h_k(z)$ is

$$e^{h_k(z)} = o(e^{h_{k+1}(z)}),$$

if $z \in G_{k+1}$ and is close to L_k , while if $z \in G_k$ and is close to L_{k-1} , then

$$e^{h_{k+1}(z)} = o(e^{h_k(z)})$$

("close" means at a small distance compared to the width of the narrower of the two tongue-shaped domains G_k and G_{k+1}).

On the curve L_k the most natural assumption is that

$$|e^{h_k(z)}| = |e^{h_{k+1}(z)}|.$$

The zeros of $F(z)$ tend asymptotically to those zeros of the function

$$e^{h_k(z)} - e^{h_{k+1}(z)},$$

that lie on the curve L_k . An asymptotic estimate for $n_k(r)$, the number of zeros of $F(z)$ lying close to the curve L_k and inside the circle $|z| < r$, is given by the formula

$$n_k(r) \sim \frac{1}{2\pi i} \{h_{k+1}(re^{i\varphi}) - h_k(re^{i\varphi})\}, \quad re^{i\varphi} \in L_k.$$

These considerations do not, of course, answer all questions relating to entire functions, however, they are necessary to an understanding of the construction of a harmonious theory of entire functions, similar to what has been done for entire functions of finite order.

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Index

- Asymptotic series, 8, 9
- Bernoulli numbers, 13, 14
- Bessel functions, 48, 168
- Borel transform, 119-125, 151
- canonical product, 72-73, 80, 90, 105, 109, 118, 161
- Euler-Maclaurin summation
 - formula, 7, 9-16, 85, 88, 137, 138, 144, 145, 153, 157
- exponential type, 119
- gamma function, 26, 35, 92-94
- generating functions, 28-41
- genus (of canonical product), 71-73
- Hadamard factorization, 105
- indicator function, 98-102, 105-110, 114, 120
- infinite order, 103
- inverse function to $x \ln x$, 2, 3
- Laplace method, 17-28, 42-45, 4, 48, 60
- Laplace transform, 33, 34, 38, 42
- maximal term (of power series), 58, 63
- Mellin transform, 33-35, 42
- order, 52
- order of growth, 52, 56, 62, 122, 124
- Phragmen-Lindelof principle, 94-98, 128
- Poisson summation formula, 137-152
- saddlepoint method, 17, 28, 32, 42-49, 65-67, 127-129
- slowly increasing function, 52-54, 56, 73, 75-76, 83
- steepest descent, path of, 43-44
- Stieltjes integral, 5
- Stirling formula, 16, 26, 48
- type (of an entire function), 52, 62
- zeta function, 15, 35

